Calculus Quiz 1

1. (5 pts)

a. Find the lowest degree of polynomial $f(x)$ such that

$$\lim_{x\to i}\frac{f(x)}{x-i}=i,\ \ i=1,2,3.$$

b. Evaluate the limit $\lim\limits_{n\to\infty} x^3\big[\sqrt{x^2+\sqrt{x^4+1}}-\sqrt{2}\,x\big]$.

Sol.

a. According to our condition, $f(x)$ is divisible by polynomial $(x-1)(x-2)(x-3)$, that is, $f(x)=(x-1)(x-2)(x-3)g(x)$ for some $g(x)$. Moreover, we have $\deg g(x)=2$ and may assume that $g(x)=ax^2+bx+c$. Then

$$\lim_{x\to1}\frac{f(x)}{x-1}=2(a+b+c)=1$$
$$\lim_{x\to2}\frac{f(x)}{x-2}=-(4a+2b+c)=2$$
$$\lim_{x\to2}\frac{f(x)}{x-3}=2(9a+3b+c)=3$$

By solving above linear system, we get $a=3,\ b=-\dfrac{23}{2},\ c=9$. Hence

$$f(x)=\frac{1}{2}(x-1)(x-2)(x-3)(6x^2-23x+18)$$

b.

$$\begin{aligned}
&\lim_{n\to\infty} x^3\big[\sqrt{x^2+\sqrt{x^4+1}}-\sqrt{2}\,x\big]\\
&=\lim_{n\to\infty}\frac{x^3\big(\sqrt{x^4+1}-x^2\big)}{\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}\,x}\\
&=\lim_{n\to\infty}\frac{x^3}{\big(\sqrt{x^2+\sqrt{x^4+1}}+\sqrt{2}\,x\big)\big(\sqrt{x^4+1}+x^2\big)}\\
&=\lim_{n\to\infty}\frac{1}{\Big(\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\Big)\Big(\sqrt{1+\frac{1}{x^4}}+1\Big)}=\frac{\sqrt{2}}{8}
\end{aligned}$$

□

2. (5 pts)

a. Consider the Fibonacci sequence $1,\ 1,\ 2,\ 3,\ 5,\ 8, \cdots$. That is, the nth term of the sequence satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-1}$$

Show that the limit of consecutive quotient $\dfrac{a_{n+1}}{a_n}$ is the golden ratio $\dfrac{1+\sqrt{5}}{2} \doteq 1.61803$.

b. Consider the sequence with recurrence relation

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n},\ \ a_1 = 3$$

Show that $\lim_{n\to\infty} a_n = \sqrt{2}$.

Proof.

a. Let $b_{n+1} = \dfrac{a_{n+1}}{a_n}$. Then

$$b_{n+1} = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{b_n}$$

Also, we have that $|b_{n+1}| \leq 1 + \dfrac{1}{|b_n|} \leq 2,\ n = 1, 2, \cdots$, so the sequence $\{b_n\}$ converges. Let $\beta = \lim_{n\to\infty} b_n$. By taking limit on both side of above equality, we get $\beta = 1 + \dfrac{1}{\beta} \Rightarrow \beta^2 - \beta - 1 = 0$. This implies that $\beta = \dfrac{1 \pm \sqrt{5}}{2}$. Note that $\beta_3 = \dfrac{a_3}{a_2} = 2 > 0$, therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \beta = \frac{1+\sqrt{5}}{2}$$

b. It is obvious $a_1^2 > 2$. For any $n \geq 2$

$$a_n^2 - 2 = \left(\frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}\right)^2 - 2 = \left(\frac{a_{n-1}^2 - 2}{2a_{n-1}}\right)^2$$

By mathematical induction, we have that $a_n^2 > 2,\ \forall n$. Thus,

$$a_n - a_{n+1} = \frac{a_n}{2} - \frac{1}{a_n} = \frac{a_n^2 - 2}{2a_n} > 0,\ \forall n$$

This implies that $\{a_n\}$ is an decreasing sequence which has lower bound $\sqrt{2}$ and hence is convergent. Let $\alpha = \lim_{n\to\infty} a_n$. By recurrence relation, we get

$$\alpha = \frac{\alpha}{2} + \frac{1}{\alpha} \Rightarrow \alpha^2 = 2 \Rightarrow \alpha = \pm\sqrt{2}$$

Since $a_n > \sqrt{2},\ \forall\, n$, so

$$\lim_{n\to\infty} a_n = \alpha = \sqrt{2}$$

□