Calculus Quiz 2

1. (5 pts) Find the limits $L = \lim_{x\to a} f(x)$ for following given functions $f(x)$ and a. And find a number $\delta > 0$ such that for all x

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

with given ε.

a. $f(x) = \sqrt{1-5x}, \quad a = -3, \quad \varepsilon = 0.5.$

b. $f(x) = \dfrac{4}{x}, \quad a = 2, \quad \varepsilon = 0.4.$

Sol.

a. It is easy to see that $\lim_{x\to -3} \sqrt{1-5x} = 4$. Since $\varepsilon = 0.5$, so

$$\begin{aligned} |f(x) - L| < \varepsilon \quad &\Leftrightarrow \quad |\sqrt{1-5x} - 4| < 0.5 \Leftrightarrow -0.5 < \sqrt{1-5x} - 4 < 0.5 \\ &\Leftrightarrow \quad 3.5 < \sqrt{1-5x} < 4.5 \Leftrightarrow 12.25 < 1 - 5x < 20.25 \\ &\Leftrightarrow \quad 11.25 < -5x < 19.25 \Leftrightarrow -3.85 < x < -2.25 \\ &\Leftrightarrow \quad -0.85 < x + 3 = x - a < 0.75 \end{aligned}$$

Choose $\delta = 0.75 > 0$, then we have that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

b. It is easy to see that $\lim_{x\to 2} \dfrac{4}{x} = 2$. Since $\varepsilon = 0.4$, so

$$\begin{aligned} |f(x) - L| < \varepsilon \quad &\Leftrightarrow \quad \left|\frac{4}{x} - 2\right| < \frac{2}{5} \Leftrightarrow -\frac{2}{5} < \frac{4}{x} - 2 < \frac{2}{5} \\ &\Leftrightarrow \quad \frac{8}{5} < \frac{4}{x} < \frac{12}{5} \Leftrightarrow \frac{5}{12} < \frac{x}{4} < \frac{5}{8} \\ &\Leftrightarrow \quad \frac{5}{3} < x < \frac{5}{2} \Leftrightarrow -\frac{1}{3} < x - 2 = x - a < \frac{1}{2} \end{aligned}$$

Choose $\delta = \dfrac{1}{3} > 0$, then we have that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

$\square$

2. (5 pts)

a. Let $f(x)$ be a function defined on $[a, b]$ and for any y between $f(a)$ and $f(b)$, there is $c \in [a, b]$ such that $f(c) = y$. Is it true that f is continuous on $[a, b]$? If not, find an counterexample.

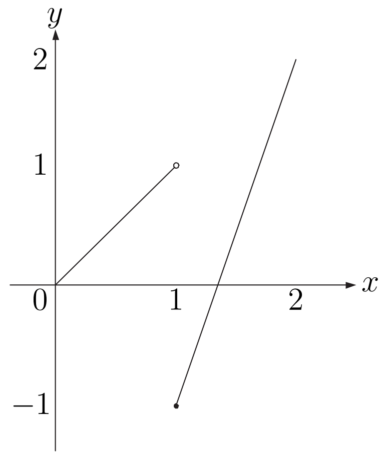
b. Suppose f is a continuous function on $[0, 1]$ and $0 \le f(x) \le 1$, $\forall x \in [0, 1]$. Show that there must exist $c \in [0, 1]$ such that $f(c) = c$.

Sol.

a. It is not true for the statement. This says that the converse of the Intermediate Value Theorem is not hold. Consider the following function defined on closed interval $[0,2]$ as

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 3x - 4, & 1 \le x \le 2 \end{cases}$$

which has graph of follows



From the figure above, it is easy to see that for any y between $f(0) = 0$ and $f(2) = 2$, there exists $c \in [0,2]$ such that $f(c) = y$. But $f(x)$ is discontinuous at $x = 1$.

b. If $f(0) = 0$ or $f(1) = 1$, we are done. If it is not the case. Since $0 \le f(x) \le 1$, $\forall x \in [0,1]$, we may assume that $f(0) = a > 0$ and $f(1) = b < 1$. Let $g(x) = f(x) - x$, then $g(x)$ is defined and continuous on $[0,1]$. Also, $g(0) = f(0) = a > 0$ and $g(1) = f(1) - 1 = b - 1 < 0$. By Intermediate Value Theorem, there exists $c \in (0,1)$ such that $g(c) = 0 \Rightarrow f(c) = c$.

$\square$