

Calculus Quiz 4

1. (5 pts)

a. Suppose $f(\frac{\pi}{3}) = 4$ and $f'(\frac{\pi}{3}) = -2$ and let

$$g(x) = f(x) \sin x, \quad h(x) = \frac{\cos x}{f(x)}$$

Find $g'(\frac{\pi}{3})$ and $h'(\frac{\pi}{3})$.

b. Let $p(x) = ax^2 + bx$, $q(x) = cx^2 + dx$ and let

$$f(x) = p(x) \cos x + q(x) \sin x$$

Determine a, b, c, d such that $f(x)$ satisfies the equation

$$f''(x) + f(x) = x \sin x$$

Sol.

a. Since $g(x) = f(x) \sin x$ and $h(x) = \frac{\cos x}{f(x)}$, then

$$\begin{aligned} g'(x) &= f'(x) \sin x + f(x) \cos x \\ h'(x) &= \frac{-f(x) \sin x - f'(x) \cos x}{f(x)^2} \end{aligned}$$

Hence

$$\begin{aligned} g'(\frac{\pi}{3}) &= f'(\frac{\pi}{3}) \sin \frac{\pi}{3} + f(\frac{\pi}{3}) \cos \frac{\pi}{3} = 2 - \sqrt{3} \\ h'(\frac{\pi}{3}) &= \frac{-f(\frac{\pi}{3}) \sin \frac{\pi}{3} - f'(\frac{\pi}{3}) \cos \frac{\pi}{3}}{f(\frac{\pi}{3})^2} = \frac{1 - 2\sqrt{3}}{16} \end{aligned}$$

b. Observe that

$$\begin{aligned} f'(x) &= p'(x) \cos x - p(x) \sin x + q'(x) \sin x + q(x) \cos x \\ &= (p'(x) + q(x)) \cos x + (q'(x) - p(x)) \sin x \\ f''(x) &= (p''(x) + q'(x)) \cos x - (p'(x) + q(x)) \sin x \\ &\quad + (q''(x) - p'(x)) \sin x + (q'(x) - p(x)) \cos x \\ &= (p''(x) + 2q'(x) - p(x)) \cos x + (q''(x) - 2p'(x) - q(x)) \sin x \end{aligned}$$

Thus

$$f''(x) + f(x) = (p''(x) + 2q'(x)) \cos x + (q''(x) - 2p'(x)) \sin x$$

Since $p(x) = ax^2 + bx$, $q(x) = cx^2 + dx$, then

$$\begin{aligned} p'(x) &= 2ax + b, & p''(x) &= 2a \\ q'(x) &= 2cx + d, & q''(x) &= 2c \end{aligned}$$

Since $f''(x) + f(x) = x \sin x$ then

$$(4cx + 2a + 2d) \cos x + (-4ax - 2b + 2c) \sin x = x \sin x$$

This implies that

$$\begin{cases} 4cx + 2a + 2d = 0 \\ -4ax - 2b + 2c = x \end{cases} \Rightarrow \begin{cases} 4c = 0 \\ 2a + 2d = 0 \\ -4a = 1 \\ -2b + 2c = 0 \end{cases}$$

$$\text{and therefore } a = -\frac{1}{4}, b = 0, c = 0, d = \frac{1}{4}.$$

□

2. (5 pts)

- a. For what values of x does the graph of $f(x) = x + 2 \sin x$ have a horizontal tangent?
- b. Find equation of the lines both tangent to parabolas $y = x^2$ and $y = -x^2 + 4x - 10$.

Sol.

- a. Since $f'(c)$ represent the slope of the tangent line of $f(x)$ at the point $x = c$, so $f(x)$ has a horizontal tangent line if and only if $f'(x) = 0$. Note that $f'(x) = 1 + 2 \cos x$, so

$$f'(x) = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow x = (2n + 1)\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

Therefore $f(x)$ has horizontal tangent at $x = (2n + 1)\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$.

- b. Denote the tangent line by L . Let $f(x) = x^2$, $g(x) = -x^2 + 4x - 10$. Suppose L tangent to $f(x)$ and the point $(x_1, f(x_1))$ and tangent to $g(x)$ at the point $(x_2, g(x_2))$. Then the equation of L can be expressed as

$$\begin{aligned} y &= f'(x_1)(x - x_1) + f(x_1) \\ &= 2x_1(x - x_1) + x_1^2 = 2x_1x - x_1^2 \\ y &= g'(x_2)(x - x_2) + g(x_2) \\ &= (-2x_2 + 4)(x - x_2) - x_2^2 + 4x_2 - 10 = (-2x_2 + 4)x + x_2^2 - 10 \end{aligned}$$

This implies that

$$\begin{aligned} \begin{cases} 2x_1 = -2x_2 + 4 \\ -x_1^2 = x_2^2 - 10 \end{cases} &\Rightarrow \begin{cases} x_1 = -x_2 + 2 \\ x_1^2 + x_2^2 = 10 \end{cases} \\ &\Rightarrow x_2^2 - 2x_2 - 3 = (x_2 + 1)(x_2 - 3) = 0 \end{aligned}$$

Hence $x_2 = -1$ or $x_2 = 3$. Therefore, the equation of the line tangent to both $y = x^2$ and $y = -x^2 + 6x - 17$ are

$$y = 6x - 9 \text{ and } y = -2x - 1$$

□