

Calculus Quiz 7

1. (5 pts) Let $f(x) = \frac{(x+1)^2}{x^2+1}$
- Find the intervals of increase or decrease.
 - Find the local maximum and minimum values.
 - Find the intervals of concavity and the inflection points.
 - Find the asymptotes.
 - Sketch the graph of $f(x)$.

Sol. Note that the denominator of $f(x)$ is always positive, so there is no vertical asymptotes of the graph of f . By evaluating the limit

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^2 + 1} = 1, \quad \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 - 2x + 1}{x^2 + 1} = 1$$

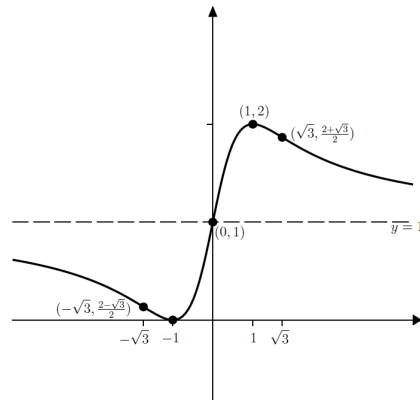
we know that there is an horizontal asymptote $y = 1$ of the graph of f . By computing the first and second derivative of f , we get

$$f'(x) = -\frac{2(x^2 - 1)}{(x^2 + 1)^2}, \quad f''(x) = \frac{4(x^3 - 3x)}{(x^2 + 1)^3}$$

It is easy to see that $f'(x) = 0 \Leftrightarrow x = \pm 1$ and $f''(x) = 0 \Leftrightarrow x = 0$ or $x = \pm\sqrt{3}$. Observe that $f''(\pm 1) = \mp 1$, thus f has local maximum at $x = 1$ and local minimum at $x = -1$. We have the following table

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f'	-	-	-	0	+	+	+	0	-	-	-
f''	-	0	+	+	+	0	-	-	-	0	+
f	\searrow	inf.pt.	\searrow	min.	\nearrow	inf.pt.	\nearrow	max.	\searrow	inf.pt.	\searrow

And the graph of f is as follows



□

2. (5 pts)

- a. Is it true that $f''(c) = 0$ implies $x = c$ is an inflection point of f ? Explain your answer.
- b. Show that for any $|\sin b - \sin a| \leq |b - a|$, $\forall a, b \in \mathbb{R}$.

Sol.

- a. It is not true that $f''(c) = 0$ implies $x = c$ is an inflection point of f . For example, consider $f(x) = x^4$, then

$$f'(x) = 4x^3, \quad f''(x) = 12x^2$$

Then $f''(0) = 0$ but $x = 0$ is not inflection point of $f(x)$.

- b. Since $\sin x$ is differentiable function defined on \mathbb{R} . For any $a, b \in \mathbb{R}$, say $a \leq b$. By Mean Value Theorem, there exists $\xi \in (a, b)$ such that

$$\sin b - \sin a = \cos \xi \cdot (b - a)$$

Since $|\cos x| \leq 1$, $\forall x$, then

$$|\sin b - \sin a| = |\cos \xi| |b - a| \leq |b - a|$$

Similarly, the argument also can apply to the case $a > b$, and hence the prove is complete.

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