Calculus Quiz 10

1. (5 pts)

a. Let $F(x) = \int_1^x \int_{\sqrt{t}}^{t^2} \frac{\sqrt{1+u^4}}{u} du dt$. Find $F''(2)$.

b. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}, \text{ for all } x > 0$$

Sol.

a. By FTC, we know that

$$F'(x) = \int_{\sqrt{x}}^{x^2} \frac{\sqrt{1+u^4}}{u} du,$$

and

$$F''(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot 2x - \frac{\sqrt{1+(\sqrt{x})^4}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{2\sqrt{1+x^8}}{x} - \frac{\sqrt{1+x^2}}{2x}$$

Hence $F''(2) = \sqrt{257} - \frac{\sqrt{5}}{4}$

b. Differentiate the equality with respect to x on both side, then by FTC, we get

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$$

Substituting $x = a$ into the equality, we have that

$$6 = 2\sqrt{a} \Rightarrow a = 9$$

□

2. (5 pts)

a. If f is continuous on $[0, \pi]$, show that

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$$

[*Hint*: Taking $u = \pi - x$]

b. Use **a.** to evaluate the integral

$$\int_0^\pi x \sin^2 x \cos^4 x dx$$

Sol.

a.

$$\begin{aligned}\int_0^{\pi} xf(\sin x)dx &= \int_{\pi}^{0}(\pi-u)f\Big(\sin(\pi-u)\Big)(-du), \text{ by letting } u{=}1{-}x{\Rightarrow}du{=}{-}dx\\ &= \int_0^{\pi}(\pi-u)f(\sin u)du = \pi\int_0^{\pi} f(\sin u)du - \int_0^{\pi} uf(\sin u)du\\ &= \pi\int_0^{\pi} f(\sin u)du - \int_0^{\pi} xf(\sin x)dx\end{aligned}$$

Thus we get $2\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx$ and therefore

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

b. By **a.**, we have that

$$\begin{aligned}\int_0^{\pi} x\sin^2 x\cos^4 x dx &= \int_0^{\pi} x\sin^2 x(1-\sin^2 x)^2 dx = \frac{\pi}{2}\int_0^{\pi}\sin^2 x(1-\sin^2 x)^2 dx\\ &= \frac{\pi}{2}\int_0^{\pi}\sin^2 x\cos^4 x dx = \frac{\pi}{2}\int_0^{\pi}(\sin x\cos x)^2\cos^2 x dx\\ &= \frac{\pi}{8}\int_0^{\pi}\sin^2 2x\Big(\frac{1+\cos 2x}{2}\Big)dx = \frac{\pi}{16}\int_0^{\pi}\sin^2 2x(1+\cos 2x)dx\\ &= \frac{\pi}{16}\int_0^{\pi}\sin^2 2x dx + \frac{\pi}{16}\int_0^{\pi}\sin^2 2x\cos 2x dx\\ &= \frac{\pi}{16}\int_0^{\pi}\frac{1-\cos 4x}{2}dx + \frac{\pi}{32}\int_0^{0} u^2 du, \text{ by letting } u{=}\sin 2x{\Rightarrow}du{=}2\cos 2x dx\\ &= \frac{\pi}{32}\int_0^{\pi} dx - \frac{\pi}{32}\int_0^{\pi}\cos 4x dx\\ &= \frac{\pi^2}{32} - \frac{\pi}{128}\int_0^{4\pi}\cos v dv, \text{ by letting } v{=}4x{\Rightarrow}dv{=}4dx\\ &= \frac{\pi^2}{32} - \frac{\pi}{128}\sin v\Big|_{v=0}^{v=4\pi} = \frac{\pi^2}{32}\end{aligned}$$

□