

Calculus Quiz 15

1. (5 pts)

- a. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x) dx$. Compare your result to the actual value. Can you explain the discrepancy?
- b. Let f be a polynomial with $\deg f = 3$ or lower, which defined on $[a, b]$. Show that the Simpson's Rule gives the exact value of $\int_a^b f(x) dx$ [Hint: it suffice to show the result when there are two subintervals ($n = 2$), since for a larger even number of subintervals the sum of exact estimates is exact.]

Sol.

- a. Let $f(x) = \cos(\pi x)$, and $\Delta x = \frac{20 - 0}{10} = 2$. The by Trapezoidal Rule,

$$\begin{aligned} T_{10} &= \frac{2}{2} [f(0) + 2(f(2) + f(4) + \cdots + f(18)) + f(20)] \\ &= \cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi \\ &= 1 + 2 \cdot 18 + 1 = 20 \end{aligned}$$

The actual value of the integral is

$$\begin{aligned} \int_0^{20} \cos(\pi x) dx &= \frac{1}{\pi} \int_0^{20\pi} \cos u du, \text{ by letting } u = \pi x \Rightarrow du = \pi dx \\ &= \frac{1}{\pi} \sin u \Big|_{u=0}^{u=20\pi} = \frac{1}{\pi} (\sin 20\pi - \sin 0) = 0 \end{aligned}$$

The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value $f(2n) = \cos 2n\pi = 1$.

- b. Let $f(x) = Ax^3 + Bx^2 + Cx + D$ and let $\Delta x = h = \frac{b-a}{2}$. Then we set $x_0 = a$, $x_1 = a + h$, $x_2 = b$. Without loss of generality, we may shift our graph of f to the left such that $x_1 = 0$, that is, by letting $y = x - a - h$ and $g(y) = f(x - a - h)$, then

$$\int_{-h}^h g(y) dy = \int_a^b f(x) dx$$

By Simpson's Rule, we have that

$$\begin{aligned} S_2 &= \frac{h}{3}(g(-h) + 4g(0) + g(h)) \\ &= \frac{h}{3}(-Ah^3 + Bh^2 - Ch + D + 4D + Ah^3 + Bh^2 + Ch + D) \\ &= \frac{2}{3}Bh^3 + 2Dh \end{aligned}$$

The exact value of integral is

$$\begin{aligned} \int_a^b f(x)dx &= \int_{-h}^h g(y)dy = \int_{-h}^h (Ay^3 + By^2 + Cy + D)dy \\ &= \left(\frac{A}{4}y^4 + \frac{B}{3}y^3 + \frac{C}{2}y^2 + Dy\right)\Big|_{-h}^h = \frac{2}{3}Bh^3 + 2Dh \end{aligned}$$

which is coincide with S_2 , the prove is complete. \square

2. (5 pts) *The extension of factorial to non-integer values.*

By the fact that the improper integral $\int_0^\infty t^{x-1}e^{-t}dt$ is convergent for $x > 0$. We define it as a function of x , called the Gamma function $\Gamma(x)$.

a. Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, in particular $\Gamma(n+1) = n!$ when n is positive integer.

b. Have known that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Find $\Gamma\left(\frac{3}{2}\right)$.

Sol.

a. For $x \geq 1$, then $x-1 \geq 0$ and thus

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt = \lim_{b \rightarrow \infty} \int_0^b t^x e^{-t} dt \\ &= \lim_{b \rightarrow \infty} \left[-t^x e^{-t} \Big|_0^b + \int_0^b x t^{x-1} e^{-t} dt \right], \text{ by letting } \begin{array}{ll} u=t^x, & dv=e^{-t}dt \\ du=xt^{x-1}dt, & v=-e^{-t} \end{array} \\ &= -\lim_{b \rightarrow \infty} b^x e^{-b} + x \lim_{b \rightarrow \infty} \int_0^b t^{x-1} e^{-t} dt \\ &= x\Gamma(x) \end{aligned}$$

where the limit $\lim_{b \rightarrow \infty} b^x e^{-b} = 0$ is due L'Hospital's Rule. In fact,

$$\begin{aligned} \lim_{b \rightarrow \infty} b^x e^{-b} &= \lim_{b \rightarrow \infty} x b^{x-1} e^{-b} = \dots = \lim_{b \rightarrow \infty} \left(\prod_{i=0}^{[x]} (x-i) \right) b^{x-[x]-1} e^{-b} \\ &= \lim_{b \rightarrow \infty} \frac{x(x-1) \cdots (x-[x])}{b^{1+[x]-x} e^b} = 0 \end{aligned}$$

For $0 < x < 1$, then $x-1 < 0$ and hence $t^{x-1} e^{-t}$ is singular at $t = 0$. Thus, similar to above argument,

$$\begin{aligned} \Gamma(x+1) &= \lim_{a \rightarrow 0} \lim_{b \rightarrow \infty} \int_a^b t^x e^{-t} dt = \lim_{a \rightarrow 0} \lim_{b \rightarrow \infty} \left[-t^x e^{-t} \Big|_a^b + x \int_a^b t^{x-1} e^{-t} dt \right] \\ &= \lim_{a \rightarrow 0} \lim_{b \rightarrow \infty} \left(a^x e^{-a} - b^x e^{-b} \right) + x \int_0^\infty t^{x-1} e^{-t} dt \\ &= \lim_{a \rightarrow 0} a^x e^{-a} - \lim_{b \rightarrow \infty} b^x e^{-b} + x\Gamma(x) = x\Gamma(x) \end{aligned}$$

In particular, for $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned} \Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \dots \\ &= n(n-1) \cdots 3 \cdot 2 \cdot \Gamma(1) = n! \int_0^\infty e^{-t} dt \\ &= n! \lim_{b \rightarrow \infty} \int_0^b e^{-t} dt = n! \lim_{b \rightarrow \infty} (1 - e^{-b}) = n!. \end{aligned}$$

b.

$$\begin{aligned} \Gamma\left(\frac{3}{2}\right) &= \int_0^\infty t^{\frac{3}{2}-1} e^{-t} dt = \int_0^\infty t^{\frac{1}{2}} e^{-t} dt \\ &= 2 \int_0^\infty s^2 e^{-s^2} ds, \text{ by letting } s=t^{\frac{1}{2}} \Rightarrow ds=\frac{1}{2}t^{-\frac{1}{2}} dt = \frac{dt}{2s} \Rightarrow 2s ds=dt \\ &= 2 \lim_{b \rightarrow \infty} \left[-\frac{1}{2} s e^{-s^2} \Big|_0^b + \frac{1}{2} \int_0^b e^{-s^2} ds \right], \text{ by letting } \begin{matrix} u=s, & dv=se^{-s^2} ds \\ du=ds, & v=-\frac{1}{2}e^{-s^2} \end{matrix} \\ &= -\lim_{b \rightarrow \infty} b e^{-b^2} + \lim_{b \rightarrow \infty} \int_0^b e^{-s^2} ds = -\lim_{b \rightarrow \infty} \frac{b}{e^{b^2}} + \int_0^\infty e^{-s^2} ds \\ &= -\lim_{b \rightarrow \infty} \frac{1}{2be^{b^2}} + \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2} \end{aligned}$$

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