Calculus Quiz 16

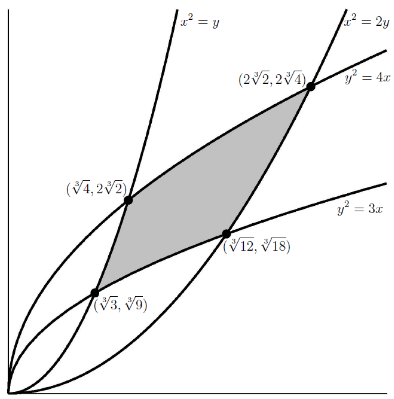
1. (5 pts)

a. Find the area of the region bounded by four parabolas $x^2 = y,\ x^2 = 2y,\ y^2 = 3x,\ y^2 = 4x$.

b. Find the total area under the curve $y = \dfrac{x\tan^{-1}x}{(1+x^2)^2}$ for $x > 0$.

Sol.

a. The region is as shown in the figure:



Let $\Gamma_1 : x^2 = y$, $\Gamma_2 : x^2 = 2y$, $\Gamma_3 : y^2 = 3x$ and $\Gamma_4 : y^2 = 4x$. It is trivial that the four parabolas have common point at the origin. We have to seek nontrivial intersection points for these parabolas. By substituting $x^2 = y$ into the equation $y^2 = 3x$, we get $x^4 - 3x = 0 \Rightarrow x(x^3 - 3) = 0 \Rightarrow x = \sqrt[3]{3}$. Thus Γ_1 and Γ_3 intersect at the point $(\sqrt[3]{3}, \sqrt[3]{9})$. Similarly, the intersections of $\Gamma_2 \cap \Gamma_3$, $\Gamma_1 \cap \Gamma_4$ and $\Gamma_1 \cap \Gamma_4$ are $(\sqrt[3]{12}, \sqrt[3]{18})$, $(\sqrt[3]{4}, 2\sqrt[3]{2})$ and $(2\sqrt[3]{2}, 2\sqrt[3]{4})$ respectively. Hence, the area A can be evaluated as

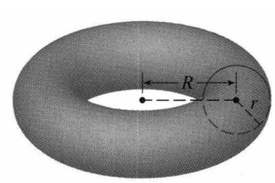
$$
\begin{aligned}
A &= \int_{\sqrt[3]{3}}^{\sqrt[3]{4}} (x^2 - \sqrt{3x})dx + \int_{\sqrt[3]{4}}^{\sqrt[3]{12}} (2\sqrt{x} - \sqrt{3x})dx + \int_{\sqrt[3]{12}}^{2\sqrt[3]{2}} \Big(2\sqrt{x} - \frac{1}{2}x^2\Big)dx \\
&= \Big(\frac{x^3}{3} - \frac{2x^{\frac{3}{2}}}{\sqrt{3}}\Big)\Big|_{\sqrt[3]{3}}^{\sqrt[3]{4}} + \Big(\frac{4x^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{\sqrt{3}}\Big)\Big|_{\sqrt[3]{4}}^{\sqrt[3]{12}} + \Big(\frac{4x^{\frac{3}{2}}}{3} - \frac{x^3}{6}\Big)\Big|_{\sqrt[3]{12}}^{2\sqrt[3]{2}} \\
&= \Big(\frac{7}{3} - \frac{4}{\sqrt{3}}\Big) + \Big(4\sqrt{3} - \frac{20}{3}\Big) + \Big(\frac{14}{3} - \frac{8}{\sqrt{3}}\Big) = \frac{1}{3}
\end{aligned}
$$

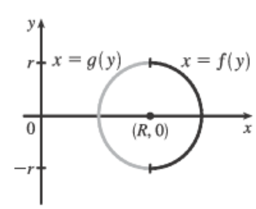
b. The total area A under the curve $y = \dfrac{x\tan^{-1}x}{(1+x^2)^2}$ for $x > 0$ is

$$
\begin{aligned}
A &= \int_0^\infty \frac{x\tan^{-1}x}{(1+x^2)^2}dx = \lim_{b\to\infty}\int_0^b \frac{x\tan^{-1}x}{(1+x^2)^2}dx \\
&= \lim_{b\to\infty}\left[\frac{-\tan^{-1}x}{2(1+x^2)}\Big|_0^b + \frac{1}{2}\int_0^b \frac{dx}{(1+x^2)^2}\right], \text{ by letting } \begin{smallmatrix} u=\tan^{-1}x, & dv=\frac{xdx}{(1+x^2)^2} \\ du=\frac{dx}{1+x^2}, & v=\frac{-1}{2(1+x^2)} \end{smallmatrix} \\
&= \lim_{b\to\infty}\left[\frac{\tan^{-1}b}{2(1+b^2)} + \frac{1}{2}\int_0^{\tan^{-1}b} \frac{\sec^2\theta}{\sec^4\theta}d\theta\right], \text{ by letting } x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta \\
&= \lim_{b\to\infty}\left[\frac{\tan^{-1}b}{2(1+b^2)} + \frac{1}{2}\int_0^{\tan^{-1}b} \cos^2\theta d\theta\right] \\
&= \lim_{b\to\infty}\left[\frac{\tan^{-1}b}{2(1+b^2)} + \frac{1}{2}\int_0^{\tan^{-1}b} \left(\frac{1+\cos 2\theta}{2}\right)d\theta\right] \\
&= \lim_{b\to\infty}\left[\frac{\tan^{-1}b}{2(1+b^2)} + \frac{1}{2}\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right)\Big|_0^{\tan^{-1}b}\right] \\
&= \lim_{b\to\infty}\left[\frac{\tan^{-1}b}{2(1+b^2)} + \frac{1}{4}\tan^{-1}b + \frac{1}{8}\sin(2\tan^{-1}b)\right] \\
&= \frac{1}{4}\cdot\frac{\pi}{2} + \frac{1}{8}\sin\pi = \frac{\pi}{8}
\end{aligned}
$$

□

2. (5 pts) We can evaluate the volume of a solid torus (the donut-shaped solid shown in the figure) by complete the following steps.





a. The torus is obtained by rotating the circle $(x-R)^2+y^2 = r^2$ about the y-axis. By slicing the torus through y, we get the cross-section $A(y)$ as annular region. Determine the cross-section function $A(y)$.

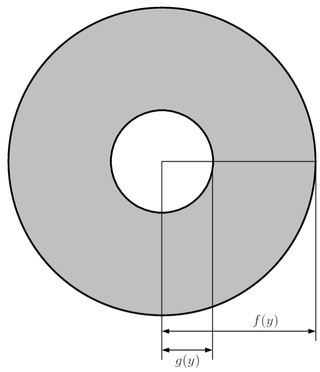
b. Evaluate the volume V as the definite integral $V = \displaystyle\int_{-r}^{r} A(y)dy$.

Sol.

a. Note that we can express the circle $(x-R)^2+y^2=r^2$ as the union of two graphs of functions $f(y)$ and $g(y)$, where

$$f(y)=R+\sqrt{r^2-y^2},\qquad g(y)=R-\sqrt{r^2-y^2}$$

For any fixed $y\in[-r,r]$, the cross section is an annular region with inner radius $g(y)$ and outer radius $f(y)$.



Thus, the area $A(y)$ of the cross section is

$$\begin{aligned}A(y)&=\pi(f(y))^2-\pi(g(y))^2=\pi\big(f(y)^2-g(y)^2\big)\\&=\pi\Big[\big(R^2+2R\sqrt{r^2-y^2}+r^2-y^2\big)-\big(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\big)\Big]\\&=4\pi R\sqrt{r^2-y^2}\end{aligned}$$

b. The volume V is

$$\begin{aligned}V&=\int_{-r}^{r}A(y)dy=4\pi R\int_{-r}^{r}\sqrt{r^2-y^2}dy=8\pi R\int_0^r\sqrt{r^2-y^2}dy\\&=8\pi r^2R\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta,\ \text{by letting } y=r\sin\theta\Rightarrow dy=r\cos\theta d\theta\\&=8\pi r^2R\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta=4\pi r^2R\Big(\int_0^{\frac{\pi}{2}}d\theta+\int_0^{\frac{\pi}{2}}\cos 2\theta d\theta\Big)\\&=4\pi r^2R\Big(\theta\Big|_0^{\frac{\pi}{2}}+\frac{1}{2}\sin 2\theta\Big|_0^{\frac{\pi}{2}}\Big)=2\pi^2r^2R\end{aligned}$$

□