Calculus Quiz 17

1. (5 pts)

a. Find the volume of solid of revolution obtained by rotating the region bounded by $y = (x^2+3x+2)^{-1}$ and $x = 0,\ x = 1$ about the x-axis and the y-axis respectively.

b. What values of p have the following property: The area of the region between the curve $y = x^{-p},\ 1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by rotating the region about the x-axis is finite.

Sol.

a. By disk method, the volume of the resulting solid is

$$V = \pi\int_0^1 \frac{dx}{(x^2+3x+2)^2} = \pi\int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$$

Assume that

$$\frac{1}{(x+1)^2(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

Then we have the following system of linear equations

$$\begin{aligned} A \quad\quad + C \quad\quad &= 0 \\ 5A + B + 4C + D &= 0 \\ 8A + 4B + 5C + 2D &= 0 \\ 4A + 4B + 2C + D &= 1 \end{aligned}$$

Thus, we get $A = -2,\ B = D = 1,\ C = 2$, and hence

$$V = \pi\int_0^1 \frac{dx}{(x+1)^2(x+2)^2} = \pi\int_0^1 \Big(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{x+2} + \frac{1}{(x+2)^2}\Big)dx$$

$$= \pi\Big(-2\int_0^1 \frac{dx}{x+1} + 2\int_0^1 \frac{dx}{x+2} + \int_0^1 \frac{dx}{(x+1)^2} + \int_0^1 \frac{dx}{(x+2)^2}\Big)$$

$$= \pi\Big(-2\ln|x+1|\Big|_0^1 + 2\ln|x+2|\Big|_0^1 + \int_1^2 u^{-2}du + \int_2^3 u^{-2}du\Big)$$

$$\text{, by letting } \begin{matrix} u=x+1 \Rightarrow du=dx \\ u=x+2 \Rightarrow du=dx \end{matrix}$$

$$= \pi\Big(-2\ln 2 + 2\ln\frac{3}{2} + \int_1^3 u^{-2}du\Big) = \pi\Big(2\ln\frac{3}{4} - u^{-1}\Big|_1^3\Big)$$

$$= \pi\Big(\ln\frac{9}{16} + \frac{2}{3}\Big)$$

b. Note that the area A of the region between $y = x^{-p}$ and the x-axis is

$$A = \int_1^\infty x^{-p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx = \lim_{b\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^b$$
$$= \lim_{b\to\infty} \left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right) = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p \le 1 \end{cases}$$

Also the volume V of the solid generated by rotating the region about the x-axis is

$$V = \pi \int_1^\infty x^{-2p} dx = \pi \lim_{b\to\infty} \int_1^b x^{-2p} dx = \pi \lim_{b\to\infty} \frac{x^{1-2p}}{1-2p}\Big|_1^b$$
$$= \pi \lim_{b\to\infty} \left(\frac{b^{1-2p}}{1-2p} - \frac{1}{1-2p}\right) = \begin{cases} \dfrac{1}{2p-1}, & \text{if } p > \dfrac{1}{2} \\ \infty, & \text{if } p \le \dfrac{1}{2} \end{cases}$$

Therefore, the curve $y = x^{-p}$ gives infinite area and finite volume for values p satisfying $\frac{1}{2} < p \le 1$.

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2. (5 pts)

a. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational filed of a planet with mass M and radius R. Use Newton's Law of Gravitation and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

b. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-\frac{Mv^2}{2RT}} dv$$

where M, R, T and v are molecular weight, gas constant, temperature, and molecular speed respectively. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

Sol.

a. By Newton's Law of Gravitation the gravitational force of the planet applying on the rocket is $F = \dfrac{GMm}{r^2}$, where r denote the distance of the rocket and the center of mass of the planet. Hence the work W needed for rocket to escape the planet is

$$W = \int_R^\infty F dr = \lim_{b\to\infty}\int_R^b \frac{GMm}{r^2}dr = -\lim_{b\to\infty} GMm\frac{1}{r}\Big|_R^b$$
$$= GMm\lim_{b\to\infty}\left(\frac{1}{R}-\frac{1}{b}\right) = \frac{GMm}{R}$$

Since the initial kinetic energy $\frac{1}{2}mv_0^2$ supplies the needed work, we have that

$$\frac{1}{2}mv_0^2 = \frac{GMm}{R} \Rightarrow v_0 = \sqrt{\frac{2GM}{R}}$$

Hence the escape velocity $v_0 = \sqrt{\frac{2GM}{R}}$.

b. Denote the constant $K = \frac{M}{2RT}$., then

$$\bar{v} = \frac{4}{\sqrt{\pi}}K^{\frac{3}{2}}\int_0^\infty v^3 e^{-Kv^2}dv = \frac{4}{\sqrt{\pi}}K^{\frac{3}{2}}\lim_{b\to\infty}\int_0^b v^3e^{-Kv^2}dv$$
$$= \frac{2}{\sqrt{\pi}}K^{\frac{3}{2}}\lim_{b\to\infty}\int_0^{b^2} we^{-Kw}dw,\ \text{by letting } w=v^2\Rightarrow dw=2vdv$$
$$= \frac{2}{\sqrt{\pi}}K^{\frac{3}{2}}\lim_{b\to\infty}\left(-\frac{w}{K}e^{-Kw}\Big|_0^{b^2}+\frac{1}{K}\int_0^{b^2}e^{-Kw}dw\right),\ \text{by letting } \begin{matrix} u=w, & dz=e^{-Kw}dw \\ du=dw, & z=\frac{-1}{K}e^{-Kw}\end{matrix}$$
$$= \sqrt{\frac{4K}{\pi}}\lim_{b\to\infty}\left(-\frac{b^2}{e^{-Kb^2}}-\frac{e^{-Kw}}{K}\Big|_0^{b^2}\right) = \sqrt{\frac{4K}{\pi}}\lim_{b\to\infty}\left(-\frac{b^2}{e^{-Kb^2}}-\frac{e^{-Kb^2}}{K}+\frac{1}{K}\right)$$
$$= \sqrt{\frac{4K}{\pi}}\left(-\lim_{b\to\infty}\frac{b^2}{e^{-Kb^2}}-\lim_{b\to\infty}\frac{e^{-Kb^2}}{K}+\frac{1}{K}\right) = \sqrt{\frac{4K}{\pi}}\left(\lim_{b\to\infty}\frac{1}{Ke^{-Kb^2}}+\frac{1}{K}\right)$$
$$= \sqrt{\frac{4}{K\pi}} = \sqrt{\frac{8RT}{\pi M}}$$

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