Calculus Quiz 18

1. (5 pts) Let $f(x) = \int_1^x \sqrt{t^2-1}dt,\ x \geq 1.$

a. Find the arc length function $s(x)$ for the curve $y = f(x)$.

b. Express the curve $y = f(x)$ as a explicit function of arc length s. That is, find the function g such that $y = g(s)$.

Sol.

a. Note that by FTC, $f'(x) = \sqrt{x^2-1}$. So the arc length function $s(x)$ for the curve $y = f(x)$ is

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1+[f'(t)]^2}dt = \int_1^x \sqrt{1+t^2-1}dt \\ &= \int_1^x t dt = \frac{t^2}{2}\Big|_1^x = \frac{x^2-1}{2} \end{aligned}$$

b. Since $s = \dfrac{x^2-1}{2}$, then $x = x(s) = \sqrt{2s+1}$. Hence the curve $y = f(x)$ can be expressed as

$$y = f(x) = f(x(s)) = f(\sqrt{2s+1}) = \int_1^{\sqrt{2s+1}} \sqrt{t^2-1}dt$$

Denote the indefinite integral $I = \int \sqrt{t^2-1}dt$, then

$$\begin{aligned} I &= \int \sec\theta \tan^2\theta d\theta, \text{ by letting } {\scriptstyle t=\sec\theta \Rightarrow dt=\sec\theta\tan\theta d\theta} \\ &= \int \sec\theta(\sec^2\theta - 1)d\theta = \int \sec^3\theta d\theta - \int \sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta - \int \sec\theta d\theta, \text{ by letting } {\scriptstyle \begin{matrix} u=\sec\theta, & dv=\sec^2\theta d\theta \\ du=\sec\theta\tan\theta d\theta, & v=\tan\theta \end{matrix}} \\ &= t\sqrt{t^2-1} - I - \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta \\ &= t\sqrt{t^2-1} - I - \int \frac{du}{u}, \text{ by letting } {\scriptstyle u=\tan\theta+\sec\theta \Rightarrow du=(\sec^2\theta+\sec\theta\tan\theta)d\theta} \\ &= t\sqrt{t^2-1} - \ln|u| - I = t\sqrt{t^2-1} - \ln|\tan\theta + \sec\theta| - I \\ &= t\sqrt{t^2-1} - \ln|t + \sqrt{t^2-1}| - I \end{aligned}$$

Therefore,

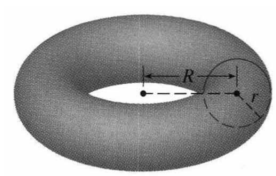
$$y = g(s) = \int_1^{\sqrt{2s+1}} \sqrt{t^2-1}dt = \frac{1}{2}t\sqrt{t^2-1} - \frac{1}{2}\ln|t+\sqrt{t^2-1}|\Big|_1^{\sqrt{2s+1}}$$
$$= \sqrt{2s+1}\sqrt{s(s+1)} - \frac{1}{2}\ln|\sqrt{2s+1}+2\sqrt{s(s+1)}|$$

□

2. (5 pts)

a. Let L be the length of the curve $y = f(x)$, $a \le x \le b$, where f is positive and has a continuous derivative. Let S_f be the surface area generated by rotating the curve about the x-axis. If c is a positive constant with $f(x) \le c$ on $[a, b]$. Let S be the surface area generated by rotating $y = f(x)$ about the line $y = c$. Express S in terms of S_f and L.

b. Find the surface area of the torus as shown in the following



Sol.

a. Let $g(x) = c - f(x)$, $x \in [a, b]$. Then the surface generated by rotating $y = f(x)$ about the line $y = c$ is equivalent to the surface generated by rotating $y = g(x)$ about the x-axis. Note that $g'(x) = -f'(x)$, thus the surface area S is

$$S = 2\pi\int_a^b g(x)\sqrt{1+[g'(x)]^2}dx$$
$$= 2\pi\int_a^b \big(c - f(x)\big)\sqrt{1+[-f'(x)]^2}dx$$
$$= 2\pi c\int_a^b \sqrt{1+[f'(x)]^2}dx - 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx$$
$$= 2\pi cL - S_f$$

b. The upper half of the torus is generated by rotating the curve $(x - R)^2 + y^2 = r^2$, $y > 0$, about the y-axis. By differentiating previous equation with respect to x, we have that $\frac{dy}{dx} = \frac{R-x}{y}$. Thus,

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(R-x)^2}{y^2} = \frac{y^2+(x-R)^2}{y^2} = \frac{r^2}{r^2-(x-R)^2}$$

Hence the surface area S is

$$\begin{aligned}
S &= 2\int_{R-r}^{R+r} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\pi r\int_{R-r}^{R+r} \frac{x}{\sqrt{r^2-(x-R)^2}}dx \\
&= 4\pi r\int_{-r}^{r} \frac{u+R}{\sqrt{r^2-u^2}}du, \text{ by letting } u=x-R \Rightarrow x=u+R,\ du=dx \\
&= 4\pi r\int_{-r}^{r} \frac{u}{\sqrt{r^2-u^2}}du + 4\pi rR\int_{-r}^{r} \frac{du}{\sqrt{r^2-u^2}} \\
&= 8\pi rR\int_{0}^{r} \frac{du}{\sqrt{r^2-u^2}}, \text{ since } \tfrac{u}{\sqrt{r^2-u^2}} \text{ is odd and } \tfrac{1}{\sqrt{r^2-u^2}} \text{ is even} \\
&= 8\pi R\int_{0}^{r} \frac{du}{\sqrt{1-\left(\frac{u}{r}\right)^2}} = 8\pi rR\int_{0}^{1} \frac{dw}{\sqrt{1-w^2}}, \text{ by letting } w=\tfrac{u}{r} \Rightarrow dw=\tfrac{du}{r} \\
&= 8\pi rR\sin^{-1} w\Big|_0^1 = 4\pi^2 rR
\end{aligned}$$

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