1. The Cartesian equation of $x = e^{2t}, y = t + 1$ is $y = (\ln x) + 1$.  
\[x = e^{2t} \Rightarrow 2t = \ln x \Rightarrow t = \frac{1}{2} \ln x. \quad y = t + 1 = \frac{1}{2} \ln x + 1.\]

2. A curve $C$ is defined by the parametric equations $x = t^2, y = t^3 - 3t$ has two horizontal tangent lines.
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0 \Rightarrow \frac{dy}{dt} = 3t^2 - 3 = 0 \Rightarrow t = \pm 1, \quad \frac{dx}{dt} = 2t \neq 0.
\]

3. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$.

4. The given curve $r = 1 + \sin \theta, r = 3 \sin \theta$ exactly has two point of intersection.

Caution: (in book p.691) The origin is also a point of intersection, but we can’t find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations.
Notice that, when represented as (0,0) or (0,π), the origin satisfies \( r = 3\sin \theta \) and so it lies on the circle; when represented as (0, 3\pi/2), it satisfies \( r = 1 + \sin \theta \) and so it lies on the cardioid. Think of two points moving along the curves as the parameter value \( \theta \) increases from 0 to 2\pi. On one curve the origin is reached at \( \theta = 0 \) and \( \theta = \pi \); on the other curve it is reached at \( \theta = 3\pi/2 \). The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

\[
\text{( ) 5. If } \lim_{n \to \infty} a_n = 0 \text{ and } \{b_n\} \text{ is bounded, then } \lim_{n \to \infty} a_n b_n = 0.
\]

<sol> T

Sol-1:

To Prove: If \( \lim_{n \to \infty} a_n = 0 \) and \( \{b_n\} \) is bounded, then \( \lim_{n \to \infty} (a_n b_n) = 0 \).

Proof: Since \( \{b_n\} \) is bounded, there is a positive number \( M \) such that \( |b_n| \leq M \) and hence, \( |a_n b_n| \leq |a_n| M \) for all \( n \geq 1 \). Let \( \varepsilon > 0 \) be given. Since \( \lim_{n \to \infty} a_n = 0 \), there is an integer \( N \) such that \( |a_n - 0| < \frac{\varepsilon}{M} \) if \( n > N \). Then

\[
|a_n b_n - 0| = |a_n b_n| = |a_n| |b_n| \leq |a_n| M = |a_n - 0| M < \frac{\varepsilon}{M} M = \varepsilon \text{ for all } n > N.
\]

Since \( \varepsilon \) was arbitrary,

\[
\lim_{n \to \infty} (a_n b_n) = 0.
\]

Sol-2:

\[
\therefore \{b_n\} \text{ is bounded, there is a positive number } M \text{ such that } |b_n| \leq M, \text{ and hence } |a_n b_n| \leq |a_n| M, \text{ for all } n \geq 1,
\]

\[
\Rightarrow 0 \leq |a_n b_n| \leq |a_n| M, \quad \lim_{n \to \infty} |a_n| = M, \quad \lim_{n \to \infty} a_n = 0 \quad (\therefore h(x) = |x| \text{ is continuous }, \forall \ x)
\]

by Squeeze theorem, we obtain \( \lim_{n \to \infty} |a_n b_n| = 0 \),

by 11.1- Thm6, we can get \( \lim_{n \to \infty} a_n b_n = 0 \).

\[
\text{( ) 6. If } \sum a_n \text{ and } \sum b_n \text{ are both divergent, then } \sum (a_n + b_n) \text{ is also divergent.}
\]

<sol> F

No. For example, take \( \sum a_n = \sum n \) and \( \sum b_n = \sum (-n) \), which both diverge, yet \( \sum (a_n + b_n) = \sum 0 \), which converges with sum 0.
7. The series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( \sin \frac{1}{n} \right) \) is divergent.

<sol> F

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1 > 0
\]

and \( \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \) converges. \( (p - \text{series}, p = \frac{3}{2} > 1) \)

8. \( \sum_{n=1}^{\infty} \left( \frac{2n + 3}{3n^2 + 2} \right)^n \) is convergent.

<sol> T

9. If \( a_n > 0 \) and \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L < 1 \), then \( \lim_{n \to \infty} a_n = 0 \).

<sol> T

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n \text{ converges (by Ratio Test)} \Rightarrow \lim_{n \to \infty} a_n = 0 \quad \text{[Theorem 11.2.6]}
\]

10. The Ratio Test can be used to determine whether \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges.

<sol> F

\[
\text{since } \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left| \frac{1}{(n+1)^3} \cdot \frac{n^3}{1} \right| = \lim_{n \to \infty} \left| \frac{n^3}{1/n^3} \right| = \lim_{n \to \infty} \frac{1}{(1+1/n)^3} = 1.
\]
1. Find the slope of the tangent to the cycloid \( x = r(\theta - \sin \theta), \ y = r(1 - \cos \theta) \) at the point where \( \theta = \frac{\pi}{3} \). Answer: ________. 

<sol> \( \sqrt{3} \) 

In the book p.670, Example 2

2. Find the exact area of the surface obtained by rotating the given curve \( x = 3t - t^3, \ y = 3t^2, \ 0 \leq t \leq 1 \) about the \( x \)-axis. Answer: ________. 

<sol> \( \frac{48}{5} \pi \)

\[ x = 3t - t^3, \ y = 3t^2, \ 0 \leq t \leq 1. \]

by the formula in book p.674,

\[ S = \int_0^1 2\pi y \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \ dt \]

\[ = \int_0^1 2\pi \cdot 3t^2 \cdot 3(1 + t^2) \ dt = 18\pi \int_0^1 (t^2 + t^4) \ dt = 18\pi \left[ \frac{1}{3}t^3 + \frac{1}{5}t^5 \right]^1_0 = \frac{48}{5} \pi \]
3. Find the first three nonzero terms in the Maclaurin series for \( e^x \sin x \). Answer: 

<sol>
\[ e^x \sin x = x + \frac{x^3}{3!} + \ldots, \] see book p.788, Example13.

4. Find the limit of \( a_n = n - \sqrt{n+1} \sqrt{n+3} \). Answer: 

<sol>
\[
\lim_{n \to \infty} a_n = \frac{-4 - 0}{1 + \sqrt{1 + 0 + 0}} = -\frac{4}{2} = -2. \text{ Converges}
\]

5. Find the values of \( x \) for which the series \( \sum_{n=0}^{\infty} (-4)^n (x-5)^n \) converges. Answer: 

<sol>
\[ \frac{19}{4} < x < \frac{21}{4} \]
6. Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 4} \) is absolutely convergent, conditionally convergent, or divergent. Answer: _________.

\[ b_n = \frac{n}{n^2 + 4} > 0 \text{ for } n \geq 1, \quad \{b_n\} \text{ is decreasing for } n \geq 2, \quad \text{and} \quad \lim_{n \to \infty} b_n = 0, \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 4} \text{ converges by the Alternating Series Test. To determine absolute convergence, choose } a_n = \frac{1}{n} \text{ to get} \]

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{n^2 + 4} = \lim_{n \to \infty} \frac{1}{n} \text{ converges by the Limit Comparison Test with the harmonic series. Thus, the series } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 + 4} \text{ is conditionally convergent.} \]

\[ \text{7. Find the exact length of the polar curve } r = \theta^3, \ 0 \leq \theta \leq 2\pi. \quad \text{Answer:} \quad \frac{8}{3} \left[ \left( \pi^2 + 1 \right)^{3/2} - 1 \right] \]

\[ L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} \ d\theta = \int_0^{2\pi} \sqrt{\theta^2 + (2\theta)^2} \ d\theta = \int_0^{2\pi} \sqrt{\theta^2 + 4\theta^2} \ d\theta \]

\[ = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} \ d\theta = \int_0^{2\pi} \frac{1}{2} \sqrt{u} \ du = \frac{1}{2} \cdot \frac{2}{3} \left[ u^{3/2} \right]_{\pi^2}^{\pi^2 + 1} = \frac{1}{2} \left[ (\pi^2 + 1)^{3/2} - 4^{3/2} \right] = \frac{8}{3} \left( \pi^2 + 1 \right)^{3/2} - 1 \]

\[ \text{8. Find parametric equations for equations for the path of a particle that moves along the circle } x^2 + (y-1)^2 = 4 \text{ three times around counterclockwise, starting at } (2, 1). \quad \text{Answer:} \quad x = 2 \cos t, \ y = 1 + 2 \sin t, \ 0 \leq t \leq 6\pi. \]

\[ \text{The circle } x^2 + (y-1)^2 = 4 \text{ has center } (0, 1) \text{ and radius 2, so by Example 4 (see book p661, 10.1) it can be represented by } x = 2 \cos t, \ y = 1 + 2 \sin t, \ 0 \leq t \leq 2\pi. \text{ This representation gives us the circle with a counterclockwise orientation starting at } (2, 1). \]

\[ \text{To get three times around in the counterclockwise direction, we use the original equations } x = 2 \cos t, \ y = 1 + 2 \sin t \text{ with the domain expanded to } 0 \leq t \leq 6\pi. \]
1. (10 points)
   a. Sketch the curve with polar equation \( r = 3 + 2 \cos \theta \) in polar coordinate.
   b. Find the area of the region enclosed by the curve of the equation \( r = 3 + 2 \cos \theta \).

   \[ \begin{array}{|c|c|c|c|c|c|c|c|c|}
   \hline
   \theta & 0 & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} & \frac{7\pi}{6} & \frac{5\pi}{4} & \frac{3\pi}{2} \\
   \hline
   r & 5 & 3+\sqrt{3} & 3+\sqrt{2} & 4 & 3 & 1 & 3-\sqrt{3} & 1 & 3 \\
   \hline
   \theta & \frac{5\pi}{3} & \frac{7\pi}{4} & \frac{11\pi}{6} & 2\pi \\
   \hline
   r & 4 & 3+\sqrt{2} & 3+\sqrt{3} & 5 \\
   \hline
   \end{array} \]

2. (10 points) Determine whether the series is convergent or divergent.
   a. \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)  
   b. \( \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2} \).

   a.

   \[
   f(x) = \frac{1}{x \ln x} \text{ is continuous and positive on } [2, \infty), \text{ and also decreasing since } f'(x) = \frac{-1 + \ln x}{x^2 (\ln x)^2} < 0 \text{ for } x > 2, \text{ so we can use the Integral Test.} \]

   \[
   \int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \left[ \ln(\ln x) \right]_{2}^{t} = \lim_{t \to \infty} \left[ \ln(\ln t) - \ln(\ln 2) \right] = \infty, \text{ so the series } \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}
   \]
b. 

\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{n^2/n} = \lim_{n \to \infty} \frac{1}{(n+1)/n}^{1/n^2} = \frac{1}{e} < 1, \text{ so the series } \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2} \text{ converges by the Root Test.} \]

3. (10 points) Find the radius of convergence and the interval of convergence of the series

\[ \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1}. \]

<sol>

If \( a_n = \frac{(x-2)^n}{n^2 + 1} \), then \( \lim_{n \to \infty} \left| \frac{an+1}{an} \right| = \lim_{n \to \infty} \frac{\frac{(x-2)^{n+1}}{(n+1)^2 + 1}}{\frac{(x-2)^n}{n^2 + 1}} = |x-2| \lim_{n \to \infty} \frac{n^2 + 1}{(n+1)^2 + 1} = |x-2|. \) By the Ratio Test, the series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1} \) converges when \( |x-2| < 1 \) \( \iff -1 < x-2 < 1 \iff 1 < x < 3. \) When \( x = 1 \), the series \( \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} \) converges by the Alternating Series Test, since \( b_n = \frac{1}{n^2 + 1} \) decreases, \( \lim_{n \to \infty} b_n = 0. \) Thus, the interval of convergence is \( I = [1, 3] \).

4. (10 points) Find a power series representation for \( f(x) = \tan^{-1} x \) by integrating a power series term by term and find its radius of convergence.

<sol>

From Example 1 in book p.771, by using Equation 1 in book p.771

\[ \frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum x^n, \quad |x| < 1, \]

we know the radius of convergence of the power series for \( \frac{1}{1+x^2} \) is 1.
Then by Example 7 in book p.774,

we can find power series representation for \( f(x) = \tan^{-1} x \).

Finally, using Theorem 2-(ii) in book p.772, we can know that

the radius of the convergence of the power series for \( \tan^{-1} x \) is also 1.
5. (10 points) For what value of $p$ is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ convergent?

$sol:$

If $p > 0$, $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ (since $\{1/n^p\}$ is decreasing) and $\lim_{n \to \infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test.

If $p \leq 0$, $\lim_{n \to \infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\iff p > 0$.

6. (10 points) Let $T_3(x)$ be the third degree Taylor polynomial for $f(x) = \sin x$ centered at $\frac{\pi}{3}$, $R_3(x)$ be the corresponding remainder term, and hence $f(x) = T_3(x) + R_3(x)$.

a. Find $T_3(x)$.

$b.$ Find an upper bound for $R_3\left(\frac{13\pi}{30}\right)$.

$sol:$

a.

b. by p.780-Thm9 Taylor's Inequality, $|R_3(x)| \leq \frac{M}{4!}\left(x - \frac{\pi}{3}\right)^4$, where $|f^{(4)}(x)| = \left|\sin x\right| \leq 1 \equiv M$, 

$$\Rightarrow \left|R_3\left(\frac{13\pi}{30}\right)\right| \leq \frac{1}{4!} \left|\frac{13\pi}{30} - \frac{\pi}{3}\right|^4 = \frac{1}{4!} \left(\frac{\pi}{10}\right)^4 = \frac{\pi^4}{4! \cdot 10^4}.$$