Calc. Homework Assignment-MGT6

陳思勉

Class: ______________________

Student Number: ______________________

Name: 李周嚴

1. Solve the first-order differential equation

$$y' = \frac{y \ln x}{x}$$

by separating variables.
[§9.2 #15]

$y' = \frac{dy}{dx} = \frac{y \ln x}{x}$

$\Rightarrow \frac{dy}{y} = \frac{\ln x}{x} dx$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{\ln x}{x} dx$

(let $u = \ln x$, $du = \frac{1}{x} dx$

$\Rightarrow \int u\, du = \frac{1}{2}u^2 + C_2 = \frac{1}{2}(\ln x)^2 + C_2$)

$\Rightarrow \ln|y| = \frac{1}{2}(\ln x)^2 + C_3$

$\Rightarrow |y| = e^{\frac{1}{2}(\ln x)^2} \cdot C_4$

$\Rightarrow y = C_5 \cdot e^{\frac{1}{2}(\ln x)^2}$ #

2. Find the solution of the initial-value problem.

$$y' = 3x^2 e^{-y}; y(0) = 1$$

[§9.2 #27]

$\frac{dy}{dx} = y' = 3x^2 e^{-y}$

$\Rightarrow e^y dy = 3x^2 dx$

$\Rightarrow \int e^y dy = \int 3x^2 dx$

$\Rightarrow e^y = x^3 + C_3 > 0$

$\Rightarrow y = \ln(x^3 + C_3)$

use the condition $y(0) = 1$

we have $1 = \ln(0 + C_3)$

$\Rightarrow C_3 = e$

hence $y = \ln(x^3 + e)$ #

3. Assume that the rate of change of the unit price of a commodity is proportional to the difference between the demand and the supply, so that

$$\frac{dp}{dt} = k(D - S)$$

where k is a constant of proportionality. Suppose that $D = 50 - 2p$, $S = 5 + 3p$, and $p(0) = 4$. Find a formula for $p(t)$.
[§9.3 #17]

$\frac{dp}{dt} = k[(50-2P) - (5+3P)] = 5k(9-P)$

$\Rightarrow \frac{1}{9-p} dp = 5k\,dt$

$\Rightarrow \int \frac{1}{9-p} dp = \int 5k\,dt$

$\Rightarrow -\ln|9-p| = 5kt + C$

$\Rightarrow |9-P| = e^{-5kt+C}$

$= e^{c} \cdot e^{-5kt}$

$\Rightarrow 9 - P = C_1 \cdot e^{-5kt}$

$\Rightarrow P = 9 - C_1 \cdot e^{-5kt}$

$\because P(0) = 4 \Rightarrow 4 = 9 - C_1 \Rightarrow C_1 =$

$\therefore P = 9 - 5e^{-5kt}$ #

4. Use Euler's method with $n = 4$ to obtain approximations to the solution of the initial-value problem when $x = 1$.

$$y' = x + y, y(0) = 1$$

[§9.4 #1]

$x_0 = 0$, $b = 1$, $n = 4$, $h = \frac{b - x_0}{n} = \frac{1-0}{4} = \frac{1}{4}$,

$\Rightarrow x_1 = x_0 + h = \frac{1}{4}$, $x_2 = x_0 + 2h = \frac{1}{2}$,

$x_3 = x_0 + 3h = \frac{3}{4}$, $x_4 = x_0 + 4h = 1 = b$

$F(x, y) = x + y$

$y_0 = y(x_0) = y(0) = 1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + \frac{1}{4}(0+1) = \frac{5}{4}$

$y_2 = y_1 + hF(x_1, y_1) = \frac{5}{4} + \frac{1}{4}\left(\frac{1}{4} + \frac{5}{4}\right) = \frac{5}{4} + \frac{3}{8} = \frac{13}{8}$

$y_3 = y_2 + hF(x_2, y_2) = \frac{13}{8} + \frac{1}{4}\left(\frac{1}{2} + \frac{13}{8}\right) = \frac{13}{8} + \frac{17}{32} = \frac{69}{32}$

$y_4 = y_3 + hF(x_3, y_3) = \frac{69}{32} + \frac{1}{4}\left(\frac{3}{4} + \frac{69}{32}\right) = \frac{69}{32} + \frac{93}{128}$

$= \frac{369}{128}$ #

$(\doteqdot 2.8828)$

5. Use Euler's method with $n = 5$ to obtain approximations to the solution to the initial-value problem

$$y' = \frac{1}{2}xy, y(0) = 1$$

over the indicated interval $0 \le x \le 1$.
[§9.4 #11]

$x_0=0,\ b=1,\ n=5,\ h=\frac{b-x_0}{n}=\frac{1-0}{5}=\frac{1}{5}$

$F(x,y)=\frac{1}{2}xy$

$\Rightarrow x_0=0,\ y_0=1$

$x_1=\frac{1}{5},\ y_1=y_0+hF(x_0,y_0)=1+\frac{1}{5}\cdot 0=1$

$x_2=\frac{2}{5},\ y_2=y_1+hF(x_1,y_1)=1+\frac{1}{5}\left(\frac{1}{2}\times\frac{1}{5}\times 1\right)=\frac{51}{50}=1.02$

$x_3=\frac{3}{5},\ y_3=y_2+hF(x_2,y_2)=1.02+\frac{1}{5}\left(\frac{1}{2}\times\frac{2}{5}\times 1.02\right)=1.0608$

$x_4=\frac{4}{5},\ y_4=y_3+hF(x_3,y_3)=1.0608+\frac{1}{5}\left(\frac{1}{2}\times\frac{3}{5}\times 1.06.08\right)=1.124448$

$x_5=1,\ y_5=y_4+hF(x_4,y_4)=1.124448+\frac{1}{5}\left(\frac{1}{2}\times\frac{4}{5}\times 1.124448\right)=1.21440384$

6. Show that the function

$$f(x) = \frac{x}{(x^2+1)^{3/2}}$$

is a probability density function on the interval $0 \le x < \infty$.
[§10.1 #9]

First, $f(x) \ge 0$ on $[0,\infty)$

$\because \int \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$ (let $u=x^2+1$. $du=2xdx$)

$=\frac{1}{2}\int u^{-\frac{3}{2}}du=\frac{1}{2}(-2)u^{-\frac{1}{2}}+C$

$=-\frac{1}{\sqrt{x^2+1}}+C$

Therefore,

$\int_0^\infty \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$

$=\lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^{\frac{3}{2}}}dx$

$=\lim_{b\to\infty}\left(-\frac{1}{\sqrt{x^2+1}}\right)\Big|_0^b=\lim_{b\to\infty}\left(-\frac{1}{\sqrt{b^2+1}}+1\right)=1$ #

7. f is the probability density function for the random variable X defined on the given interval. Find the indicated probabilities.

$$f(x) = \frac{1}{4\sqrt{x}}; [1, 9]$$

a. $P(X \ge 4)$ **b.** $P(1 \le X < 8)$
c. $P(X = 3)$ **d.** $P(X \le 4)$
[§10.1 #23]

a. $P(X\ge4)=\int_4^9\frac{1}{4}x^{-\frac{1}{2}}dx$

$=\frac{1}{2}x^{\frac{1}{2}}\Big|_4^9=\frac{1}{2}(3-2)=\frac{1}{2}$ #

b. $P(1\le X<8)=\int_1^8\frac{1}{4}x^{-\frac{1}{2}}dx$

$=\frac{1}{2}x^{\frac{1}{2}}\Big|_1^8=\frac{1}{2}(2\sqrt{2}-1)\doteqdot 0.9142$ #

c. $P(X=3)=\int_3^3\frac{1}{4}x^{-\frac{1}{2}}dx=\frac{1}{2}x^{\frac{1}{2}}\Big|_3^3=0$ #

d. $P(X\le4)=\int_1^4\frac{1}{4}x^{-\frac{1}{2}}dx$

$=\frac{1}{2}x^{\frac{1}{2}}\Big|_1^4=\frac{1}{2}(2-1)=\frac{1}{2}$ #

8. The life span (in years) of a certain brand of plasma TV is a continuous random variable with probability density function

$$f(t) = 9(9 + t^2)^{-3/2} \qquad (0 \le t < \infty)$$

How long is one of these plasma TVs expected to last?
[§10.2 #21]

Let T be the life span of a certain brand of plasma TV.

$E(T)=\int_0^\infty t\cdot 9(9+t^2)^{-\frac{3}{2}}dt$

$=\lim_{b\to\infty}\frac{-9}{\sqrt{9+t^2}}\Big|_0^b$

$=\lim_{b\to\infty}\left[\frac{-9}{\sqrt{9+b^2}}+3\right]$

$=3$

$\int t\cdot 9(9+t^2)^{-\frac{3}{2}}dt$

let $u=t^2+9$, $du=2tdt$

$=\int\frac{9}{2}u^{-\frac{3}{2}}du$

$=-9u^{-\frac{1}{2}}+C$

$=\frac{-9}{\sqrt{9+t^2}}+C$

hence the plasma TVs are expected to last for 3 years.