Calculus Homework Assignment 1

Class: ______________________

Student Number: ______________________

Name: [illegible]

饒瑞彬

1. Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

a. $a_n = \dfrac{\sin n}{n}$

b. $a_n = \dfrac{n!}{n^n}$ (*Hint*: Compare with $\frac{1}{n}$)

[§10.1 #45, 63]

a. converges

$\because |\sin n| \le 1 \Rightarrow -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty}(-\frac{1}{n}) = 0$, $\lim_{n\to\infty}\frac{1}{n} = 0$.

by the Sandwich Thm, $\lim_{n\to\infty} a_n = 0$ #

b. converges

$0 < \frac{n!}{n^n} = \frac{n(n-1)\cdots 2\cdot 1}{n\cdot n\cdots n\cdot n} < \frac{1}{n}$. By Sandwich Thm,

$0 \le \lim_{n\to\infty}\frac{n!}{n} \le \lim_{n\to\infty}\frac{1}{n} = 0 \quad \therefore \lim_{n\to\infty} a_n = 0$ #

2. Assume that the sequence converges and find its limit.

$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots\ldots$ [§10.1 #98]

$a_1 = \sqrt{1}$

$a_2 = \sqrt{1+\sqrt{1}} = \sqrt{1+a_1}$

$a_3 = \sqrt{1+\sqrt{1+\sqrt{1}}} = \sqrt{1+a_2}$

$\vdots$

$a_n = \sqrt{1+a_{n-1}}$

$a_{n+1} = \sqrt{1+a_n}$, $n \ge 1$

$\because$ the sequence converges

$\therefore$ let $\lim_{n\to\infty} a_n = L$.

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sqrt{1+a_n}$ $\because f(x)=\sqrt{1+x}$ is continuous

Thm 10.1.3 $\Rightarrow L = \sqrt{1+L}$ at $x = L \ge 1$. $\therefore f(a_n) \to f(L)$

$\Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1\pm\sqrt{5}}{2}$ (負不合 $\because a_n > 0$)

$\therefore \lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$ #

3. Determine if the series converges or diverges. If a series converges, find its sum.

a. $\sum_{n=1}^{\infty}(\ln\sqrt{n+1} - \ln\sqrt{n})$

b. $\sum_{n=1}^{\infty}\dfrac{2n+1}{n^2(n+1)^2}$ [§10.2 #37, 44]

a.

$S_k = (\cancel{\ln\sqrt{2}} - \ln\sqrt{1}) + (\cancel{\ln\sqrt{3}} - \cancel{\ln\sqrt{2}}) + \cdots + (\cancel{\ln\sqrt{k}} - \cancel{\ln\sqrt{k-1}})$

$+ (\ln\sqrt{k+1} - \cancel{\ln\sqrt{k}})$ $\quad S_k \equiv \sum_{n=1}^{k}(\ln\sqrt{n+1} - \ln\sqrt{n})$

$= \ln\sqrt{k+1} - \ln\sqrt{1} = \ln\sqrt{k+1}$

$\Rightarrow \lim_{k\to\infty} S_k = \lim_{k\to\infty}\ln\sqrt{k+1} = \infty \quad \therefore$ diverges #

b. $S_k = \sum_{n=1}^{k}(2n+1)/[n^2(n+1)^2]$

$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow S_k = (\frac{1}{1} - \cancel{\frac{1}{4}}) + (\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}) + \cdots + (\cancel{\frac{1}{(k-1)^2}} - \cancel{\frac{1}{k^2}})$

$+ (\cancel{\frac{1}{k^2}} - \frac{1}{(k+1)^2}) = 1 - \frac{1}{(k+1)^2}$

$\Rightarrow \lim_{k\to\infty} S_k = \lim_{k\to\infty}(1 - \frac{1}{(k+1)^2}) = 1$ #

4. Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

a. $\sum_{n=0}^{\infty}\dfrac{\cos n\pi}{5^n}$

b. $\sum_{n=1}^{\infty}\ln\left(\dfrac{n}{2n+1}\right)$ [§10.2 #54, 66]

a. geometric series $a = 1$, $r = -\frac{1}{5}$

$\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n} = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots$

$= \frac{1}{1-(-\frac{1}{5})} = \frac{5}{6} \Rightarrow$ series converges

b.

$\sum_{n=1}^{\infty}\ln(\frac{n}{2n+1})$

$\because \lim_{n\to\infty}\ln(\frac{n}{2n+1}) = \ln(\frac{1}{2}) \ne 0$.

$\therefore$ series diverges by the nth-Term Test for Divergence.

(Over Please)

5. Which of the series converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

a. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

b. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ [§10.3 #1, 35]

$f(x)=\frac{1}{x^2}$ continuous.

a. converges. positive, decreasing for all $x \geq 1$.

$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$

$= \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{-1}{b}+1\right)=1$

By Integral Test $\because \int_1^\infty \frac{1}{x^2}dx$ converges $\Rightarrow \sum_{n=1}^\infty \frac{1}{n^2}$ converges

b. converges. $f(x)=e^x/(1+e^{2x})$

$\int_1^\infty \frac{e^x}{1+e^{2x}}dx = \int_e^\infty \frac{1}{1+u^2}du = \lim_{b\to\infty}(\tan^{-1}u)\Big|_e^b$

$u=e^x$ $\Rightarrow du=e^x dx$

$f(x)>0$ continuous decreasing. $x\geq 1$.

$=\lim_{b\to\infty}(\tan^{-1}b-\tan^{-1}e)=\frac{\pi}{2}-\tan^{-1}e$ converges

Integral Test $\sum_{n=1}^\infty \frac{e^n}{1+e^{2n}}$ converges ※

6. Use the Integral Test to show that the series

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges. [§10.3 #58]

Since $0 \leq e^{-x^2} \leq e^{-x}$ $\forall x\geq 1$ ($\because x^2 \geq x$, $-x^2 \leq -x$ $\forall x \geq 1$, e^x increasing)

and $\int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]\Big|_1^b$

$=\lim_{b\to\infty}(-e^{-b}+e^{-1})=e^{-1}$ conv.

$\therefore$ By Thm 8.8.2,

$\int_1^\infty e^{-x^2}dx$ converges $\Rightarrow \sum_{n=1}^\infty e^{-n^2}$ conv.

Integral Test $f(x)=e^{-x^2}>0$, continuous decreasing

$\Rightarrow \sum_{n=0}^\infty e^{-n^2} = 1+\underbrace{\sum_{n=1}^\infty e^{-n^2}}_{\text{converges}}$

$\therefore \sum_{n=0}^\infty e^{-n^2}$ converges ※

7. Determine if the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{\frac{3}{2}}}$

b. $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

(*Hint* : Limit Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

[§10.4 #5, 16]

a. $\because \sum_{n=1}^\infty \frac{1}{n^{\frac{3}{2}}}$ is a p-series and $p=\frac{3}{2}>1$

$\therefore \sum_{n=1}^\infty \frac{1}{n^{\frac{3}{2}}}$ converges

$\because 0 \leq \cos^2 n \leq 1$ $\therefore \frac{\cos^2 n}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}$ $\forall n\geq 1$

$\Rightarrow$ By Comparison Test, $\sum_{n=1}^\infty \frac{\cos^2 n}{n^{\frac{3}{2}}}$ converge

b. $\because \sum_{n=1}^\infty \frac{1}{n^2}$ converges ($p=2>1$) p series

$\Rightarrow \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n^2}}\left(\frac{-2}{n^3}\right)}{\left(\frac{-2}{n^3}\right)} = 1>0$

$\therefore$ By Limit Comparison Test, $\sum_{n=1}^\infty \ln(1+\frac{1}{n^2})$ converge

8. Which of the series converge, and which diverge? Use any method, and give reasons for your answers.

a. $\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$

b. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ [§10.4 #35, 45]

a. $\sum_{n=1}^\infty \frac{1-n}{n2^n} = \underbrace{\sum_{n=1}^\infty \frac{1}{n2^n}}_{①} + \underbrace{\sum_{n=1}^\infty \frac{-n}{n2^n}}_{②}$

① $\because \frac{1}{n2^n} \leq \frac{1}{2^n}$, $n\geq 1$, $\sum_{n=1}^\infty \frac{1}{2^n} = \frac{\frac{1}{2}}{1-(\frac{1}{2})}=1$ converges

$\therefore \sum_{n=1}^\infty \frac{1}{n2^n}$ converges by comparison Test.

② $\sum_{n=1}^\infty \frac{-1}{2^n}$ converges (geometric series $r=\frac{1}{2}$, $|r|<1$)

Thus $\sum_{n=1}^\infty \frac{1-n}{n2^n}$ converges by Thm 10.2.8(1)

b. $\because \sum_{n=1}^\infty \frac{1}{n}$ diverges (p-series $\wedge$ $P=1$)

$\Rightarrow$ By Limit Comparison Test,

$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0}\frac{\sin x}{x} = 1 > 0 \Rightarrow \sum_{n=1}^\infty \sin\frac{1}{n}$ diverges ※