Calculus Homework Assignment 2

Class: ____________________

Student Number: ____________________

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1. Determine if each series converges absolutely or diverges.

a. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$

b. $\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$ [§10.5 #3, 12]

use Ratio test, we have

$$\lim_{n\to\infty}\left|\frac{n!}{(n+2)^2}\cdot\frac{(n+1)^2}{(n-1)!}\right| = \lim_{n\to\infty}\frac{n(n+1)^2}{(n+2)^2} = \lim_{n\to\infty}\frac{n^3+2n^2+n}{n^2+4n+n} \to \infty > 1$$

so, series diverges.

use Root test, we have

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(-\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}\right|} = \lim_{n\to\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+\frac{1}{n}} = \ln(e^2) = 2 > 1$$

so, series diverges.

2. Use any method to determine if the series converges or diverges. Give reasons for your answer.

a. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

b. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ [§10.5 #39, 43]

use Root test, we have

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{-n}{(\ln n)^n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{\ln n} = \frac{\lim_{n\to\infty}\sqrt[n]{n}}{\lim_{n\to\infty}\ln n} = 0 < 1$$

so, series converges.

use Ratio test, we have

$$\lim_{n\to\infty}\left|\frac{((n+1)!)^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$$

so, series converges.

3. Which of the series converge, and which diverge? Give reasons for your answers.

a. $\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

b. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot(2n-1)}{4^n 2^n n!}$ [§10.5 #59, 61]

a. use Root test, we have

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^n}{2^{n^2}}\right|} = \lim_{n\to\infty}\frac{n}{2^n} = \lim_{n\to\infty}\frac{1}{2^n\ln 2} = 0 < 1$$

so, series converges.

b. use Ratio test, we have

$$\lim_{n\to\infty}\left|\frac{1\cdot 3\cdots(2n-1)(2n+1)}{4^{n+1}2^{n+1}(n+1)\,n!}\cdot\frac{4^n 2^n n!}{1\cdot 3\cdots(2n-1)}\right| = \lim_{n\to\infty}\frac{2n+1}{8(n+1)} = \frac{1}{4} < 1$$

so, series converges.

4. Determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$

b. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{2^n}{n^2}$ [§10.6 #1, 7]

a. let $a_n = \frac{1}{\sqrt{n}}$

① $a_n > 0$

② $a_n > a_{n+1} \Rightarrow \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$

③ $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

By Alternating series test we know that series converges.

b.

$\because \lim_{n\to\infty}\frac{2^n}{n^2} = \infty$

$\Rightarrow \lim_{n\to\infty}(-1)^{n+1}\frac{2^n}{n^2}$ does not exist.

$\therefore$ Diverges. (By nth item test for Divergence).

5. Determine if the series converges absolutely, converges or diverges? Give reasons for your answers.

$$\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$$

[§10.6 #41]

$\boxed{1}$ $\sum_{n=1}^{\infty} |(-1)^n(\sqrt{n+1}-\sqrt{n})| = \sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n}$

$= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + \sqrt{n+1}-\sqrt{n} + \cdots$

when $n \to \infty$, series will diverges.

$\Rightarrow$ not converges absolutely

$\boxed{2}$ Let $a_n = \sqrt{n+1}-\sqrt{n}$

① $a_n > 0$

② $\frac{d a_n}{dn} = \frac{1}{2}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) < 0 \Rightarrow a_n > a_{n+1}$

③ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}-1}{\frac{1}{\sqrt{n}}} = 0 \Rightarrow$ converges.

$\boxed{3}$ $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$ is convergence conditionally.

6. Determine if the series converges absolutely, converges or diverges? Give reasons for your answers.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$

[§10.6 #44]

$\boxed{1}$ $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{(\sqrt{n+1}+\sqrt{n})\cdot(\sqrt{n+1}-\sqrt{n})}$

$= \sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n} = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + \cdots + \sqrt{n+1}-\sqrt{n} + \cdots$

when $n\to\infty$, series will diverges $\Rightarrow$ not converges absolutely.

$\boxed{2}$ Let $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1}-\sqrt{n}$

① $a_n > 0$

② $\frac{d}{dn}a_n = \frac{1}{2}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) < 0 \Rightarrow a_n > a_{n+1}$

③ $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}-1}{\frac{1}{\sqrt{n}}} = 0$

so series converges.

$\boxed{3}$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ is convergence conditionally #

7. (a) Find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}$$

[§10.7 #27]

$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x+2)^{n+1}}{(n+1)\cdot 2^{n+1}}\cdot\frac{n\cdot 2^n}{(-1)^{n+1}(x+2)^n}\right|$ (By Ratio test)

$= \lim_{n\to\infty}\left|\frac{(x+2)\cdot n}{2n+2}\right| = \lim_{n\to\infty}\left|\frac{x+2}{2+\frac{2}{n}}\right| = \left|\frac{x+2}{2}\right| < 1.$

$\Rightarrow$ radius is 2, interval of converge absolutely is $-4 < x < 0$

Now consider the end points. $x=0$, $x=-4$:

$$\begin{cases} x=0, \ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ is converge (by alternating test)} \\ x=-4, \ \sum_{n=1}^{\infty}\frac{-2^n}{n} \text{ is diverge } \left(\because \lim_{n\to\infty}\frac{-2^n}{n}\to\infty\right)\end{cases}$$

(a) the radius is 2, the interval of convergence is $-4 < x \le 0$

(b) the interval of absolutely convergence is $-4 < x < 0$

(c) the series converges conditionally at $x=0$.

8. Use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

$$\sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4^n}$$

[§10.7 #43]

Consider $\sum_{n=0}^{\infty}\frac{x^n}{4^n}$ and use Ratio test.

$\Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}}\cdot\frac{4^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{4}\right| = \left|\frac{x}{4}\right| < 1$ $(R=4)$.

Let $f(x) = (x-1)^2$

$\sum_{n=0}^{\infty}\frac{[f(x)]^n}{4^n}$ is converges absolutely for

$|(x-1)^2| < 4 \Rightarrow -1 < x < 3.$

Now consider endpoints $x=-1$, $x=3$.

$$\begin{cases} x=-1, \ \sum_{n=0}^{\infty} 1 \text{ is diverge} \\ x=3, \ \sum_{n=0}^{\infty} 1 \text{ is diverge}\end{cases}$$

$\therefore$ interval of convergence is $-1 < x < 3$.

then

$\because \sum_{n=0}^{\infty}\frac{x^n}{4^n} = \frac{1}{1-\frac{x}{4}}$

$\therefore \sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4^n} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{-x^2+2x+3}$