Calculus Homework Assignment 3

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1. Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

$$f(x) = \ln x, \qquad a = 1$$

[§10.8 #3] $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$. So $f(1) = 0$, $f'(1) = 1$, $f''(1) = 1$

$f'''(1) = 2$, Now $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

So $P_0(x) = 0$, $P_1(x) = (x-1)$

$P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$

$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

2. Find the Taylor series generated by f at $x = a$.

$$f(x) = 2^x, \qquad a = 1$$

[§10.8 #30]

$f(x) = 2^x$, $f'(x) = 2^x \cdot \ln 2$, $f''(x) = 2^x \cdot (\ln 2)^2$

$\dots$, $f^{(n)}(x) = 2^x \cdot (\ln 2)^n$

So $f(1) = 2$, $f'(1) = 2 \cdot \ln 2$, $f''(2) = 2 \cdot (\ln 2)^2$,

$\dots$, $f^{(n)}(1) = 2 \cdot (\ln 2)^n$

The Taylor series generated by f at $x = a = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 2 + 2\ln 2\,(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \cdots + \frac{2(\ln 2)^n}{n!}(x-1)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

3. Find the first three nonzero terms of the Maclaurin series for each function and the values of x for which the series converges absolutely.

$$f(x) = (\sin x)\ln(1+x)$$

[§10.8 #35]

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$, $|x| < \infty$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, $|x| < 1$

$$f(x) = \left(x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \cdots\right) + \left(-\frac{1}{2}x^3 + \frac{1}{2\cdot 3!}x^5 - \frac{1}{2\cdot 5!}x^7 + \cdots\right) + \left(\frac{1}{3}x^4 - \frac{1}{3\cdot 3!}x^6 + \frac{1}{3\cdot 5!}x^8 - \cdots\right)$$

$\Rightarrow f(x) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 + \cdots$, $|x| < 1$

4. Use power series operations to find the Taylor series at $x = 0$ for the function.

$$\frac{x}{3}\ln(1+x^2)$$

[§10.9 #27]

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, $|x| < 1$

$\Rightarrow \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots$, $|x| < 1$

$\frac{x}{3}\ln(1+x^2) = \frac{x^3}{3} - \frac{x^5}{6} + \frac{x^7}{9} - \cdots$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{3k}, \quad |x| < 1$$

5. Find the first four nonzero terms in the Maclaurin series for the function.

$$e^x \sin x$$

[§10.9 #29]

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ on $(-\infty, \infty)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ on $(-\infty, \infty)$

$e^x \sin x = \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right) +$

$\left(-\frac{x^3}{3!} - \frac{x^4}{3!} - \frac{x^5}{2\cdot 3!} - \frac{x^6}{3!\cdot 3!} - \cdots\right) +$

$\left(\frac{x^5}{5!} + \frac{x^6}{5!} + \frac{x^7}{2\cdot 5!} + \frac{x^8}{3!\cdot 5!} + \cdots\right) + \cdots$

$= x + x^2 + \frac{1}{3}x^3 - \left(\frac{1}{5!} + \frac{1}{4!} - \frac{1}{2\cdot 3!}\right)x^5 - \cdots$

$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \cdots$ on $(-\infty, \infty)$

6. Find the first four terms of the binomial series for the functions.

a. $(1 + x)^{\frac{1}{3}}$

b. $\left(1 + \frac{x}{2}\right)^{-2}$

[§10.10 #2, 5]

a) $\binom{\frac{1}{3}}{1} = \frac{1}{3}$, $\binom{\frac{1}{3}}{2} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} = -\frac{1}{9}$

$\binom{\frac{1}{3}}{3} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} = \frac{5}{81}$

$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \cdots$

$|x| < 1$

b) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$, $|x| < 1$

$\Rightarrow -\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \cdots)$, $|x| < 1$

$\Rightarrow \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$, $|x| < 1$

$\left(1 + \frac{x}{2}\right)^{-2} = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \cdots$, $|x| < 2$

7. Find parametric equations for the semicircle

$$x^2 + y^2 = a^2, \qquad y > 0$$

using as parameter the slope $t = dy/dx$ of the tangent to the curve at (x, y). [§11.1 #29]

$2x + 2y\frac{dy}{dx} = \frac{d}{dx}(a^2) = 0$

So $\frac{dy}{dx} = -\frac{x}{y} = t$, $y > 0$

By assumption, we have

$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{a^2}{y^2}$

So $y^2 = \frac{a^2}{t^2+1}$ & $y = \frac{|a|}{\sqrt{t^2+1}}$

Thus, $y = \frac{|a|}{\sqrt{t^2+1}}$ and $x = -yt = \frac{-|a|t}{\sqrt{t^2+1}}$

8. Find the point on the parabola

$$x = t, \qquad y = t^2, \qquad -\infty < t < \infty,$$

closest to the point $(2, 1/2)$.

(Hint: Minimize the square of the distance as a function of t.) [§11.1 #39]

Let $d = \sqrt{(x-2)^2 + (y - \frac{1}{2})^2}$ the distance fcn.

Then $d = d(t) = \sqrt{(t-2)^2 + (t^2 - \frac{1}{2})^2}$

and $d^2 = (t-2)^2 + (t^2 - \frac{1}{2})^2$

Now $(d^2)' = 2(t-2) + 2(t^2 - \frac{1}{2})\cdot 2t$

$= 2t - 4 + 4t^3 - 2t$

$= 4t^3 - 4 = 4(t-1)(t^2+t+1)$

$(d^2)'' = 12t^2$

So $(d^2)''\big|_{t=1} = 12 > 0$

Thus, $(x, y) = (1, 1^2) = (1, 1)$ is the pt.