Calculus Quiz 2 ATMO

1. (4 *point*) Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{3}t)\mathbf{i} + (\sqrt{6}t)\mathbf{j} + (2 - \frac{3}{2}t^2)\mathbf{k}$$

from $(0,0,2)$ to $(\sqrt{3}, \sqrt{6}, \frac{1}{2})$.

Sol. By solving $\mathbf{r}(t) = (0,0,2)$ and $\mathbf{r}(t) = (\sqrt{3}, \sqrt{6}, \frac{1}{2})$, it is not hard to see that $\mathbf{r}(0) = (0,0,2)$ and $\mathbf{r}(1) = (\sqrt{3}, \sqrt{6}, \frac{1}{2})$. Also,

$$\mathbf{v}(t) = \sqrt{3}\,\mathbf{i} + \sqrt{6}\,\mathbf{j} - (3t)\mathbf{k}$$

Hence the arc length of $\mathbf{r}(t)$ form $(0,0,2)$ to $(\sqrt{3}, \sqrt{6}, \frac{1}{2})$ is

$$L = \int_0^1 |\mathbf{v}|dt = \int_0^1 \sqrt{9 + 9t^2}dt = 3\int_0^1 \sqrt{1+t^2}dt$$

Note that

$$\begin{aligned}\int \sqrt{1+t^2}dt &= \int \sec\theta \cdot \sec^2\theta d\theta, \text{ by letting } t = \tan\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta d\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{2}\left(t\sqrt{1+t^2} + \ln|t + \sqrt{1+t^2}|\right) + C\end{aligned}$$

Therefore,

$$\begin{aligned}L &= 3\int_0^1 \sqrt{1+t^2}dt = \frac{3}{2}\left[t\sqrt{1+t^2} + \ln|t+\sqrt{1+t^2}|\right]_0^1 \\ &= \frac{3}{2}(\sqrt{2} + \ln(1+\sqrt{2}))\end{aligned}$$

□

2. (6 *point*) Find $\mathbf{T}, \mathbf{N}$, and κ for the space curve

$$\mathbf{r}(t) = (\sqrt{2}e^t\cos t)\mathbf{i} + (\sqrt{2}e^t\sin t)\mathbf{j} + \sqrt{2}\,\mathbf{k}$$

Sol. Observe that

$$\mathbf{v}(t) = \left(\sqrt{2}e^t(\cos t - \sin t)\right)\mathbf{i} + \left(\sqrt{2}e^t(\cos t + \sin t)\right)\mathbf{j}$$

So we have that

$$|\mathbf{v}| = \sqrt{2e^{2t}(\cos t - \sin t)^2 + 2e^{2t}(\cos t + \sin t)^2} = 2e^t$$

and thus

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\cos t - \sin t}{\sqrt{2}}\,\mathbf{i} + \frac{\cos t + \sin t}{\sqrt{2}}\,\mathbf{j}$$

Also, $\frac{d\mathbf{T}}{dt} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\cos t - \sin t}{\sqrt{2}}\,\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\,\mathbf{j}$, which implies

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1$$

Hence,

$$\begin{aligned} \mathbf{N} &= \frac{\frac{d\mathbf{T}}{dt}}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-\cos t - \sin t}{\sqrt{2}}\,\mathbf{i} + \frac{\cos t - \sin t}{\sqrt{2}}\,\mathbf{j} \\ \kappa &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2}e^{-t} \end{aligned}$$

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