Calculus Quiz 9 ATMO

1. (5 *point*) Integrate $G(x,y,z) = y^2$ over the sphere $x^2+y^2+z^2 = 4$.
Sol. Consider the spherical coordinate, then the sphere $S : x^2 + y^2 + z^2 = 4$ can be parametrized as

$$S : \mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k},\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

Then

$$\begin{aligned} \mathbf{r}_\phi &= (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k} \\ \mathbf{r}_\theta &= (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \end{aligned}$$

and

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (2\sin 2\phi)\mathbf{k} \\ \Rightarrow\ |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 4\sin^2 2\phi} = 4\sin\phi \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_S G(x,y,z)d\sigma &= \iint_S y^2 d\sigma = \int_0^{2\pi}\int_0^{\pi} 4\sin^2\phi\sin^2\theta|\mathbf{r}_\phi \times \mathbf{r}_\theta|d\phi d\theta \\ &= 16\int_0^{2\pi}\int_0^{\pi} \sin^2\phi\sin^2\theta\sin\phi d\phi d\theta = 16\int_0^{\pi}(1-\cos^2\phi)\sin\phi d\phi \int_0^{2\pi}\sin^2\theta d\theta \\ &= 16\int_1^{-1}(z^2-1)dz\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta = 16\left[\frac{z^3}{3} - z\right]_1^{-1}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{64\pi}{3} \end{aligned}$$

□

2. (5 *point*) Evaluate $\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} d\sigma$, where S is the hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0$.
Sol. Let $f(x,y,z) = x^2 + y^2 + z^2$, then the hemisphere is the level surface $f(x,y,z) = 1,\ z \ge 0$. Then

$$\begin{aligned} \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &\Rightarrow |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2 \\ &\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \end{aligned}$$

Also we have that

$$\nabla \times (y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}$$

Since $z \geq 0$, so $z = \sqrt{1 - x^2 - y^2}$. Thus the hemisphere is defined over the region $R = \{(x, y) \big| x^2 + y^2 \leq 1\}$. By taking $\mathbf{p} = \mathbf{k}$, then

$$\begin{aligned} \iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} d\sigma &= \iint_R (-\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA \\ &= -\iint_R z \frac{1}{z} dA = -\int_0^{2\pi} \int_0^1 r dr d\theta = -\int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^1 d\theta \\ &= -\frac{1}{2} \int_0^{2\pi} d\theta = -\pi \end{aligned}$$

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