微積分聯合教學 (物理、化材、地科、大氣)　期末考試題卷　**June 23, 2005**

考試時間 100 分鐘，請盡量依照題號順序將答案寫在答案卷上，不必抄題。試題卷有四面，共 8 大題。答案卷務必記得寫學號、姓名，試題卷不必繳回。考試開始 20 分鐘後不得入場，開始 40 分鐘前不得離場。爲維持機會之平等，考試期間禁止使用字典、計算機及任何通訊器材。

**1.** (*20* points)　是非題，請答 **T** (True) 或 **F** (False)

1.1 Let $f(x,y)$ be any continuous function on $[0,1]\times[0,2]$. Then the following identity must be *always* true:

$$\int_0^2\int_0^1 f(x,y)\,dx\,dy=\int_0^1\int_0^2 f(x,y)\,dy\,dx$$

1.2 The following inequality is true:

$$\int_0^1\int_0^1 \frac{1}{1+x^2+y^2}\,dx\,dy<\frac{\pi}{4}$$

1.3 A thin, flat plate covers a region $R$ on the $xy$-plane. The plate's density at the point $(x,y)$ is $\delta(x,y)$, and its center of mass is at $(\bar{x},\bar{y})$. Then $(\bar{x},\bar{y})$ must be an *interior point* of $R$.

1.4 If $f(x,y)$ is continuous over $R_1$ and $R_2$, and $\iint_{R_1} dA=\iint_{R_2} dA$, then $\iint_{R_1} f(x,y)\,dA=\iint_{R_2} f(x,y)\,dA$.

1.5 The vector field $\mathbf{F}=(2x-3)\,\mathbf{i}-z\,\mathbf{j}+\cos x\,\mathbf{k}$ is conservative.

1.6 The differential form $y\,dx+x\,dy+4\,dz$ is exact.

1.7 Let $\mathbf{F}$ satisfy $\nabla\times\mathbf{F}=\mathbf{0}$ on a region $D$ in space, then $\mathbf{F}$ is a conservative vector field.

1.8 There exists a vector field $\mathbf{F}$ with twice-differentiable components and $\mathbf{curl}\,\mathbf{F}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$.

1.9 The outward flux of a vector field $\mathbf{F}=M\,\mathbf{i}+N\,\mathbf{j}$ across a simple closed curve $C$ equals the double integral of $(\mathbf{curl}\mathbf{F})\cdot\mathbf{k}$ over the region $R$ enclosed by $C$:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds=\iint_R\Big(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\Big)\,dx\,dy$$

(後面還有)

1.10 If the necessary partial derivatives of the components of the field $\mathbf{G} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ are continuous, then $\mathbf{curl}(\mathrm{div}\mathbf{G}) = \mathrm{div}(\mathbf{curl}\mathbf{G}) = 0$.

**2.** (*20* points) 選擇題，皆單選，請用大寫字母 **A**, **B**, **C** 或 **D** 答題

2.1 Which of the following is a potential function $f(x, y, z)$ for the vector field

$$\mathbf{F}(x, y, z) = \cos x \sin y\,\mathbf{i} + \sin x \cos y\,\mathbf{j} + \mathbf{k}\,?$$

(A) $\sin x \sin y$ (B) $\cos x \cos y$ (C) $\sin x \sin y + z$ (D) $\cos x \cos y + z$

2.2 令 $D$ 是空間中直立的斜面圓柱體。它的底面是 $xy$ 平面、頂面是 $z = 4 - y$ 平面、底面的圓心在點 $(0, 1, 0)$ 而半徑為 1。以下何者不是求 D 之體積的積分算式？

(A) $\displaystyle\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-y} r\,dz\,dr\,d\theta$

(B) $\displaystyle\int_{-1}^{1} 12\sqrt{1-x^2}\,dx$

(C) $\displaystyle\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_0^{1} 4 - y\,dz\,dy\,dx$

(D) $\displaystyle\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} 4 - y\,dy\,dx$

2.3 Which of the following integrals is equivalent to

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{1 + x^2 + y^2}\,dx\,dy\,?$$

(A) $\displaystyle\int_0^{\pi}\int_0^{\infty} \frac{1}{1+r^2}\,dr\,d\theta$

(B) $\displaystyle\int_0^{\pi/2}\int_0^{\infty} \frac{1}{1+r^2}\,dr\,d\theta$

(C) $\displaystyle\int_0^{\pi}\int_0^{\infty} \frac{r}{1+r^2}\,dr\,d\theta$

(D) $\displaystyle\int_0^{\pi/2}\int_0^{\infty} \frac{r}{1+r^2}\,dr\,d\theta$

2.4 What is the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$?

(A) $\dfrac{32}{3}$ (B) $\dfrac{16}{3}$ (C) $\dfrac{8}{3}$ (D) $\dfrac{4}{3}$

(後面還有)

2.5 What is the value of the following triple integral in cylindrical coordinates?

$$\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$$

(A) $\frac{18}{5}\pi$ (B) $\frac{6}{5}\pi$ (C) $\frac{3}{10}\pi$ (D) $\frac{9}{10}\pi$

2.6 Let $S$ be the surface parametrized by $\mathbf{r}(\theta, z) = 3\sin 2\theta\,\mathbf{i} + 6\sin^2\theta\,\mathbf{j} + z\,\mathbf{k}$ where $0 \le \theta \le \pi$. Which of the following is equivalent to the surface differential $d\sigma$?

(A) $d\theta\,dz$ (B) $3\,d\theta\,dz$ (C) $6\,d\theta\,dz$ (D) $12\,d\theta\,dz$

2.7 Evaluate $\int_C x + y\,ds$ where $C$ is the straight line segment $x = t$, $y = 1 - t$ and $z = 0$ from $(0, 1, 0)$ to $(1, 0, 0)$. Which is the answer?

(A) 0 (B) 1 (C) $\sqrt{2}$ (D) $\sqrt{3}$

2.8 Let $M$, $N$ and $P$ be functions of $(x, y, z)$. When $M\,dx + N\,dy + P\,dz$ is an exact differential form, which of the following statements is *not* true?

(A) There is a function $f(x, y, z)$ such that $df = M\,dx + N\,dy + P\,dz$.

(B) Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, then $\mathbf{F}$ is a conservative field.

(C) $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 0$ where $\mathbf{F}$ is as defined in (B) and $S$ is an orientable smooth surface.

(D) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 0$ where $\mathbf{F}$ is as defined in (B) and $C$ is a simple closed curve.

2.9 Find the circulation of the vector field $\mathbf{F} = (x - y)\,\mathbf{i} + x\,\mathbf{j}$ around the circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \le t < 2\pi$. Which is the answer?

(A) $\pi$ (B) $2\pi$ (C) 1 (D) 2

2.10 Let $\mathbf{F}$ be the gravitational field $\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Which of the following statements is *not* true?

(A) $\text{div}\mathbf{F} = 0$

(B) $\iiint_D \nabla\cdot\mathbf{F}\,dV = 0$ where $D$ is any solid that does not contain the origin.

(C) Let $S$ be *any* smooth surface with *outwrad* unit normal vector $\mathbf{n}$ that encloses the origin $(0, 0, 0)$, then $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 4\pi$.

(D) There is a continuously differentiable vector field $\mathbf{H}$ such that $\nabla\times\mathbf{H} = \mathbf{F}$.

(後面還有)

**3.** (*10* points) Evaluate the following integrals

$$\text{(a)} \int_0^3 a^x\,dx, \quad a \neq 0 \quad \text{(2 points)} \qquad \text{(b)} \int_0^1 \frac{x^3-1}{\ln x}\,dx \quad \text{(8 points)}$$

**4.** (*10* points) Let $R$ be the region in the first quadrant bounded by the $x$-axis, the parabola $y^2 = 2x$ and the line $x + y = 4$. Find the area of $R$, then find the *centroid* of $R$.

**5.** (*10* points) 令 $u = 3x + 2y$ 和 $v = x + 4y$，求 Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)}$$

並且利用這種變數變換，做以下 double integral

$$\iint_R \left(3x^2 + 14xy + 8y^2\right) dx\,dy$$

其中 $R$ 是平面上由 $3x + 2y = 2$, $3x + 2y = 6$, $x + 4y = 0$ 和 $x + 4y = 4$ 這四條直線圍成的平行四邊形區域。(這一題也許不需要變數變換也能求解。但是，為了考試，請按照指示使用變數變換步驟。)

**6.** (*10* points) (a) Show that if $R$ is the region in the plane bounded by a piecewise smooth simple closed curve $C$, then

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

(b) Use the formula above to find the area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

**7.** (*10* points) Apply Green's Theorem to evaluate the integral $\oint_C y^2\,dx + x^2\,dy$ where $C$ is the boundary of the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

**8.** (*10* points) Find the circulation of the field $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ around the closed semicircular path that consists of the semicircle $\mathbf{r}_1(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, for $0 \leq t \leq \pi$, followed by the line segment $\mathbf{r}_2(t) = t\,\mathbf{i}$, for $-a \leq t \leq a$.