Calculus Final Examination Jan 8, 2007

1. (20 points)是非題，請答 T (True) 或 F (False)
   1.1 $\frac{1}{24} x^{10} - \frac{1}{2} x^{6} + x^{2}$ is the Taylor polynomial of degree 10 at $x = 0$ for $x^{2} \cos x^{2}$.
   1.2 We can use the trigonometric substitution $x = 2 \tan \theta$ to evaluate the definite integral as follows: $\int_{0}^{2} \frac{1}{(x^{2} + 4)^{2}} \, dx = \int_{0}^{2} \frac{1}{8 \sec^{2} \theta} \, d\theta$.
   1.3 If $\sum_{n=1}^{\infty} a_{n}$ and $\sum_{n=1}^{\infty} b_{n}$ are divergent series, then $\sum_{n=1}^{\infty} (a_{n} + b_{n})$ is also divergent.
   1.4 If $a_{n} \geq 0$ and $\frac{a_{n+1}}{a_{n}} < 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_{n}$ is convergent.
   1.5 The radius of convergence for the series $\sum_{n=0}^{\infty} \left(\frac{x^{2} + 1}{3}\right)^{n}$ is $\sqrt{2}$.
   1.6 $\frac{d}{dx} \int_{1}^{x^{2}+3} \sqrt{1 + \sin t^{2}} \, dt = \sqrt{1 + \sin(x^{2} + 3)^{2}}$.
   1.7 If $\lim_{n \to \infty} a_{n} = 0$, then $\sum_{n=1}^{\infty} a_{n}$ converges.
   1.8 Let $f(x) = 3^{x}$, then $f'(x) = x 3^{x-1}$.
   1.9 $\int_{0}^{2} \frac{1}{(x - 1)^{2}} \, dx = -2$.
   1.10 $\lim_{x \to \infty} \frac{\ln x}{x} = 0$.

2. (20 points) 選擇題，皆單選，請用大寫字母 A, B, C 或 D 答題
   2.1 $\lim_{n \to \infty} \sqrt[n]{n} =$
   
   (A) $\infty$  (B) 1  (C) $e^{-1}$  (D) $e$

   (背面還有)
2.2 Which of the following can not be the interval of convergence for the power series \( \sum_{n=0}^{\infty} a_n x^n \)?

(A) \((-\infty, \infty)\)  
(B) \(\{0\}\)  
(C) \([0, 1]\)  
(D) \([-\frac{1}{2}, \frac{1}{2}]\)

2.3 What is the interval of convergence of \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} x^n \)?

(A) \([-3, 3]\)  
(B) \((-3, 3]\)  
(C) \((-1, 1]\)  
(D) \([-1, 1)\)

2.4 Which of the following series is convergent?

(A) \(\sum_{n=1}^{\infty} \frac{n}{n + 1}\)  
(B) \(\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4 + 1}}\)  
(C) \(\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}\)  
(D) \(\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}\)

2.5 If \(f'(c) = 0\), which of the following statements may not be true?

(A) \(c\) is a critical point of \(f(x)\).
(B) \(f(x)\) has a local maximum or local minimum at \(x = c\).
(C) \(\lim_{x \to c} f(x) = f(c)\).
(D) If \(y = ax + b\) is the tangent line of \(f(x)\) at \((c, f(c))\), then \(a = 0\).

2.6 The Taylor’s Theorem states that if \(f(x)\) is differentiable through order \(n + 1\) in an open interval \(I\) containing \(a\), then for each \(x \in I\)

\[
f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)
\]

where the remainder

(A) \(R_n(x) = \frac{f^{(n+1)}(a)}{(n + 1)!}(x - a)^{n+1}\)
(B) \(R_n(x) = \frac{f^{(n+1)}(a)}{(n + 1)!}(x - c)^{n+1}\) for some number \(c\) between \(a\) and \(x\)
(C) \(R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}\) for some number \(c\) between \(a\) and \(x\)
(D) \(R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - c)^{n+1}\) for some number \(c\) between \(a\) and \(x\)
2.7 Which of the following improper integrals is convergent?

(A) \( \int_1^\infty \frac{\sin^2 x}{x^2} \, dx \)  \hspace{1cm} (B) \( \int_2^\infty \frac{1}{\ln x} \, dx \)

(C) \( \int_1^\infty e^{x^2} \, dx \)  \hspace{1cm} (D) \( \int_2^\infty \frac{x}{\sqrt{x^4 - 1}} \, dx \)

2.8 Which of the following functions describes the decay of a radioactive element.

(A) \( y = y_0 \ln kt, \ k > 0 \)  \hspace{1cm} (B) \( y = y_0 \ln kt, \ k < 0 \)

(C) \( y = y_0 e^{kt}, \ k > 0 \)  \hspace{1cm} (D) \( y = y_0 e^{kt}, \ k < 0 \)

2.9 \( \int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{4 - 9x^2}} = \)

(A) \( \left[ \frac{2}{3} \sin^{-1} \frac{2x}{3} \right]_0^{1/\sqrt{3}} \)  \hspace{1cm} (B) \( \left[ \frac{1}{3} \sin^{-1} \frac{3x}{2} \right]_0^{1/\sqrt{3}} \)

(C) \( \left[ \frac{3}{2} \sin^{-1} \frac{2x}{3} \right]_0^{1/\sqrt{3}} \)  \hspace{1cm} (D) \( \left[ \frac{1}{2} \sin^{-1} \frac{3x}{2} \right]_0^{1/\sqrt{3}} \)

2.10 Which of the following series diverges?

(A) \( \sum_{n=1}^\infty \frac{1}{n^3 - n - 1} \)  \hspace{1cm} (B) \( \sum_{n=1}^\infty (1 - \frac{1}{n})^n \)

(C) \( \sum_{n=1}^\infty \frac{2^n}{n!} \)  \hspace{1cm} (D) \( \sum_{n=1}^\infty \frac{(-1)^{n+1} (2n - 1)}{n^3 + 1} \)

3. (20 points) 填充題

3.1 Let \( f(x) = e^{2x} \) and \( p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \) be the polynomial of degree 4 such that \( p(0) = f(0) \) and the derivatives satisfy \( p^{(i)}(0) = f^{(i)}(0) \) for each \( i = 1, 2, 3, 4 \). Find \( a_0 + a_1 + a_2 + a_3 + a_4 \).

3.2 The Maclaurin series for \( \sin^2 x \) is \( x^2 - \frac{1}{3} x^4 + a_5 x^5 + a_6 x^6 + \cdots \). Find \( a_6 \).

3.3 Find \( f'(x) \) for

\[
f(x) = \int_0^{\tan^{-1} x^2} \frac{1}{1 + t^2} \, dt.
\]

3.4 Find \( \int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta \).

3.5 Find \( \int_0^1 \frac{6x + 7}{(x + 2)^2} \).

(背面還有)
4. (10 points) Let \(|x|\) be the largest integer \(\leq x\).
   (a) Sketch the graph of \(y = |x|\) for \(x \in [-2, 2]\).
   (b) For what values of \(x\) is \(|x|\) differentiable?
   (c) Find \(\int_{-2}^{2} |x| \, dx\).
   (d) Find \(\lim_{x \to 3/2} \frac{|x|}{x}\).
   (e) Find \(\lim_{x \to 0} \frac{|x|}{x}\).

5. (10 points) Find the length of the curve
   \[ y = \ln(\sec x), \quad 0 \leq x \leq \frac{\pi}{4}. \]

6. (10 points) (a) Prove that
   \[ \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (6/10) \]
   (b) Find \(\int \sin^5 x \, dx\). (4/10)

7. (10 points) Expand \(f(x) = \frac{1}{(1 + x^2)^3}\) in power of \(x\) and determine the interval of convergence of this series.

8. (10 points) (a) Write the Maclaurin series for \(\cos x\) and prove that the series converges to \(\cos x\). (5/10)
   (b) Use the Maclaurin series to estimate \(\int_0^1 \cos x^2 \, dx\) with an error of magnitude less than 0.001. (5/10)

9. (10 points) Find \(f''(0)\) for
   \[ f(x) = (2x^4 + 2)^{3x}. \]