Calculus　　Final Examination　　Jan 8, 2007

考試時間 120 分鐘，是非、選擇、填充題請在專用答案卷上作答，其他題目請盡量依照題號順序寫在考試卷上，不必抄題。試題共有 2 張, 4 面, 9 大題, 120 分。答案卷務必記得寫學號、姓名，試題不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

1. (*20* points) 是非題，請答 **T** (True) 或 **F** (False)

1.1 $\frac{1}{24}x^{10} - \frac{1}{2}x^6 + x^2$ is the Taylor polynomial of degree 10 at $x = 0$ for $x^2 \cos x^2$.

1.2 We can use the trigonometric substitution $x = 2\tan\theta$ to evaluate the definite integral as follows: $\int_0^2 \frac{1}{(x^2+4)^2}\,dx = \int_0^2 \frac{1}{8\sec^2\theta}\,d\theta$.

1.3 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series, then $\sum_{n=1}^{\infty}(a_n + b_n)$ is also divergent.

1.4 If $a_n \geq 0$ and $\frac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

1.5 The radius of convergence for the series $\sum_{n=0}^{\infty}(\frac{x^2+1}{3})^n$ is $\sqrt{2}$.

1.6 $\frac{d}{dx}\int_1^{x^2+3} \sqrt{1+\sin t^2}\,dt = \sqrt{1+\sin(x^2+3)^2}$.

1.7 If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

1.8 Let $f(x) = 3^x$, then $f'(x) = x3^{x-1}$.

1.9 $\int_0^2 \frac{1}{(x-1)^2}\,dx = -2$.

1.10 $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

2. (*20* points) 選擇題，皆單選，請用大寫字母 **A**, **B**, **C** 或 **D** 答題

2.1 $\lim_{n\to\infty} \sqrt[n]{n} =$

(A) ∞　(B) 1　(C) e^{-1}　(D) e

(背面還有)

2.2 Which of the following can **not** be the *interval of convergence* for the power series $\sum_{n=0}^{\infty} a_n x^n$?

(A) $(-\infty, \infty)$ (B) $\{0\}$ (C) $[0, 1]$ (D) $[-\frac{1}{2}, \frac{1}{2}]$

2.3 What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} x^n$?

(A) $[-3, 3]$ (B) $(-3, 3]$ (C) $(-1, 1)$ (D) $[-1, 1)$

2.4 Which of the following series is convergent?

(A) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ (B) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ (C) $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$ (D) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

2.5 If $f'(c) = 0$, which of the following statements may **not** be true?

(A) c is a critical point of $f(x)$.

(B) $f(x)$ has a local maximum or local minimum at $x = c$.

(C) $\lim_{x \to c} f(x) = f(c)$.

(D) If $y = ax + b$ is the tangent line of $f(x)$ at $(c, f(c))$, then $a = 0$.

2.6 The *Taylor's Theorem* states that if $f(x)$ is differentiable through order $n+1$ in an open interval I containing a, then for each $x \in I$

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the *remainder*

(A) $R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$

(B) $R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-c)^{n+1}$ for some number c between a and x

(C) $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some number c between a and x

(D) $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$ for some number c between a and x

(後面還有)

2.7 Which of the following improper integrals is convergent?

(A) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ (B) $\displaystyle\int_2^{\infty} \frac{1}{\ln x}\,dx$

(C) $\displaystyle\int_1^{\infty} e^{x^2}\,dx$ (D) $\displaystyle\int_2^{\infty} \frac{x}{\sqrt{x^4-1}}\,dx$

2.8 Which of the following functions describes the decay of a radioactive element.

(A) $y = y_0 \ln kt,\ k > 0$ (B) $y = y_0 \ln kt,\ k < 0$

(C) $y = y_0 e^{kt},\ k > 0$ (D) $y = y_0 e^{kt},\ k < 0$

2.9 $\displaystyle\int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{4-9x^2}} =$

(A) $\left[\frac{2}{3}\sin^{-1}\frac{2x}{3}\right]_0^{1/\sqrt{3}}$ (B) $\left[\frac{1}{3}\sin^{-1}\frac{3x}{2}\right]_0^{1/\sqrt{3}}$

(C) $\left[\frac{3}{2}\sin^{-1}\frac{2x}{3}\right]_0^{1/\sqrt{3}}$ (D) $\left[\frac{1}{2}\sin^{-1}\frac{3x}{2}\right]_0^{1/\sqrt{3}}$

2.10 Which of the following series diverges?

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-n-1}$ (B) $\displaystyle\sum_{n=1}^{\infty} (1-\frac{1}{n})^n$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (D) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^3+1}$

3. (*20* points) 填充題

3.1 Let $f(x) = e^{2x}$ and $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be the polynomial of degree 4 such that $p(0) = f(0)$ and the derivatives satisfy $p^{(i)}(0) = f^{(i)}(0)$ for each $i = 1, 2, 3, 4$. Find $a_0 + a_1 + a_2 + a_3 + a_4$.

3.2 The Maclaurin series for $\sin^2 x$ is $x^2 - \frac{1}{3}x^4 + a_5x^5 + a_6x^6 + \cdots$. Find a_6.

3.3 Find $f'(x)$ for

$$f(x) = \int_0^{\tan^{-1} x^2} \frac{1}{1+t^2}\,dt.$$

3.4 Find $\displaystyle\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta.$

3.5 Find $\displaystyle\int_0^{1} \frac{6x+7}{(x+2)^2}.$

(背面還有)

4. (*10* points) Let $\lfloor x \rfloor$ be the largest integer $\leq x$.

(a) Sketch the graph of $y = \lfloor x \rfloor$ for $x \in [-2, 2)$.

(b) For what values of x is $\lfloor x \rfloor$ differentiable?

(c) Find $\displaystyle\int_{-2}^{2} \lfloor x \rfloor \, dx$.

(d) Find $\displaystyle\lim_{x \to 3/2} \frac{\lfloor x \rfloor}{x}$.

(e) Find $\displaystyle\lim_{x \to 0} \frac{\lfloor x \rfloor}{x}$.

5. (*10* points) Find the length of the curve

$$y = \ln(\sec x), \quad 0 \leq x \leq \frac{\pi}{4}.$$

6. (*10* points) **(a)** Prove that

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \qquad (6/10)$$

(b) Find $\displaystyle\int \sin^5 \, dx$. (4/10)

7. (*10* points) Expand $f(x) = \dfrac{1}{(1+x^2)^3}$ in power of x and determine the interval of convergence of this series.

8. (*10* points) **(a)** Write the Maclaurin series for $\cos x$ and prove that the series converges to $\cos x$. (5/10)

(b) Use the Maclaurin series to estimate $\displaystyle\int_0^1 \cos x^2 \, dx$ with an error of magnitude less than 0.001. (5/10)

9. (*10* points) Find $f'(0)$ for

$$f(x) = (2x^4 + 2)^{3^x}.$$