Calculus　　Final Examination　　June 26, 2007

考試時間 120 分鐘，是非、選擇、填充題請在專用答案卷上作答，其他題目請盡量依照題號順序寫在考試卷上，不必抄題。試題共有 2 張, 4 面, 7大題, 總分 121 分。答案卷務必記得寫學號、姓名，試題不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

1. (*15* points) 是非題，請答 **T** (True) 或 **F** (False)

1.1 Since $f(x,y,z) = x + 2y + 3z$ is continuous and $D = \{(x,y,z) | x^2 + y^2 + z^2 = 25\}$ is closed and bounded, f assumes global maximum and global minimum on D.

1.2 Let D denote the region bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$. If $(\rho, \phi, \theta) \in D$, then $0 \le \theta \le 2\pi$, $0 \le \phi \le \dfrac{\pi}{4}$, $0 \le \rho \le 2\cos\phi$.

1.3 Let $\mathbf{G} = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + zyz^2\mathbf{k}$. Then $\iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, d\sigma = 0$ for any closed oriented surface in the space whose outward unit normal field is $\mathbf{n}$.

1.4 If the necessary partial derivatives of the components of the field $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ are continuous, then $\text{div}(\nabla \times \mathbf{G}) = \nabla \cdot (\nabla \times \mathbf{G}) = 0$.

1.5 There exists a vector field $\mathbf{F}$ with twice-differentiable components and $\text{curl}\,\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

2. (*21* points) 選擇題，皆單選，請用大寫字母 **A**, **B**, **C** 或 **D** 答題

2.1 $\displaystyle\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \sin(x^2+y^2)\, dx\, dy$ is *not* equal to

(A) $\displaystyle\int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} \sin(x^2+y^2)\, dy\, dx$

(B) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^a r\sin r^2\, dr\, d\theta$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^a \sin r^2\, dr\, d\theta$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \int_0^a r\sin r^2\, dr\, d\theta$

(背面還有)

2.2 Consider the function $f(x, y) = y^3 - 3xy + x^3$ and the set $D = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 2\}$. Which statement listed below is *not* true?

(A) $\boldsymbol{\nabla} f(x, y) = (-3y + 3x^2, 3y^2 - 3x)$

(B) $\boldsymbol{\nabla} f(x, y) = 0 \Longleftrightarrow (x, y) = (0, 0)$ or $(1, 1)$

(C) If $f(a, b) \geq f(x, y), (a, b), (x, y) \in D$, then (a, b) lies on the boundary of D.

(D) If $f(a, b) \leq f(x, y), (a, b), (x, y) \in D$, then (a, b) is an interior point of D.

2.3 Let C be a curve joining a point P to a point Q in the $xyz-$space. Let $-C$ denote the curve consisting the same points as C but with the reverse direction. Let $f(x, y, z)$ be a scalar function and let $\mathbf{F}(x, y, z)$ be a vector field. Then

(A) $\displaystyle\int_{-C} f\, ds = \int_C f\, ds$ (B) $\displaystyle\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$

(C) $\displaystyle\int_{-C} f\, dx = \int_C f\, dx$ (D) $\displaystyle\int_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$

2.4 Which of the following regions in the $xy-$plane is simply connected?

(A) $\{(x, y) \mid x^2 + y^2 > 0\}$ (B) $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$

(C) $\{(x, y) \mid x \neq 0 \cup y > 0\}$ (D) $\{(x, y) \mid x^2 + y^2 > 1\}$

2.5 Let C be a counterclock simple closed curve in the $xy-$plane, and R be the region bounded by C. Which of the following is *not* equal to the area of R?

(A) $\displaystyle\iint_R 1\, dA$ (B) $\displaystyle\oint_C y\, dx$ (C) $\displaystyle\oint_C x\, dy$ (D) $\displaystyle\frac{1}{2}\oint_C -y\, dx + x\, dy$

2.6 Value of the line integral $\int_C \sqrt{x^2 + y^2}\, ds$ along the curve $\mathbf{r}(t) = 4\cos t\, \mathbf{i} + 4\sin t\, \mathbf{j} + 3t\, \mathbf{k}$, $-2\pi \leq t \leq 2\pi$ is

(A) 80π (B) 320π (C) 400π (D) 1600π

2.7 Let $\mathbf{F} = yz\, \mathbf{i} + xz\, \mathbf{j} + xy\, \mathbf{k} = \boldsymbol{\nabla}(xyz)$, Which of the following may *not* be true?

(A) $\mathbf{F}$ is a conservative vector field.

(B) $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

(C) $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$ for any simple smooth closed-loop C in the space.

(D) $yz\, dx + xz\, dy + xy\, dz$ is an exact differential form.

3. (*25* points) 填充題

3.1 The value of $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ is ______________.

3.2 Set up the integral with the order $dz\,dr\,d\theta$ for evaluating the triple integral of the function $F(x,y,z)=\cos(z^3)$ over the solid region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the plane $z=1$. ______________. **(Do not evaluate the integral.)**

3.3 The curl of $\mathbf{F}=(x^2-y)\mathbf{i}+4z\,\mathbf{j}+x^2\mathbf{k}$ is ______________.

3.4 Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ for the vector field $\mathbf{F}=x^2\mathbf{i}-y\,\mathbf{j}$ along the curve $x=y^2$ from $(4,2)$ to $(1,-1)$. Answer: ______________

3.5 Find the surface area of the surface $x^2-2\ln x+\sqrt{15}\,y-z=0$ above the square R: $1\le x\le 2$, $0\le y\le 1$ in the xy-plane. Answer : ______________

4. (*15* points) Use the divergence theorem to compute the total flux of $\mathbf{F}(x,y,z)=x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$ out of the solid cylinder: $x^2+y^2\le 4$, $0\le z\le 3$, including the top and base.

5. (*15* points) Let $\mathbf{F}(x,y)=\dfrac{-y}{x^2+y^2}\mathbf{i}+\dfrac{x}{x^2+y^2}\mathbf{j}$ for $x>0, y>0$. Find $f(x,y)$ with $\nabla f=\mathbf{F}$

6. (*18* points) Let surface S be the cone

$$z=\sqrt{x^2+y^2},\quad 0\le z\le 1.$$

(a) (*3 points*) Find a parametrization of S.

(b) (*5 points*) Find the surface area of S by the parametrization you found in part (a).

(c) (*3 points*) Let C be the boundary of S: $x^2+y^2=1$, $z=1$. Let C be oriented counterclockwise as viewed from above. With this orientation, find the unit normal vector $\mathbf{n}$ for S.

(d) (*7 points*) Find the circulation of the field $\mathbf{F}=(x^2-y)\mathbf{i}+4z\mathbf{j}+x^2\mathbf{k}$ around the curve C as described in part (c).You can use the result of Problem 3.3.

7. (*12* points) Find the work done by $\mathbf{F} = (4x - 2y)\mathbf{i} + (2x - 4y)\mathbf{j}$ in moving a particle once counterclockwise around the curve C: $(x-2)^2 + (y-2)^2 = 4$.