微積分聯合教學　段考六　June 17, 2008

考試時間 120 分鐘，試題有兩張紙，共四面，滿分 120 分。所有題目都請在考試卷上作答，而是非與填充題必須寫在第一頁。考試卷務必寫學號、姓名，試題不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

是非題 (*15* points)，請答 **T** (True) 或 **F** (False)

1. There is a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ at P(1,-1,1) equals -3 (that is, $D_u T(1, -1, 1) = -3$).

2. $\displaystyle\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy = \int_0^{2\pi} \int_0^\infty \frac{r}{(1 + r^2)^2}\, dr\, d\theta$

3. Let D be an open connected region in space. If $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed loop in D, then $\mathbf{F}$ must be a conservative vector field in D.

4. There is a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ whose components M, N and P are twice differentiable such that $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

5. $\displaystyle\lim_{n\to\infty} (-1)^n \big(1 - \frac{1}{n}\big) = \pm 1$

填充題 (*50* points)，$\boxed{\text{A}}$-$\boxed{\text{J}}$ 每格 5 分

1. Find the linearization $L(x, y)$ of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point $(3, 2)$. $\boxed{\text{A}}$

2. Let s be the arc length parameter of the plane curve $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$ for $\pi/2 \le t \le \pi$. Find the derivative ds/dt. $\boxed{\text{B}}$

(背面還有)

3. Evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{x\, e^{2y}}{4-y}\, dy\, dx$$

You may have to change the order of the integrals. $\boxed{\text{C}}$

4. Find a potential function for the following field. $\boxed{\text{D}}$

$$\mathbf{F} = 2\cos y\,\mathbf{i} + (\frac{1}{y} - 2x\sin y)\,\mathbf{j} + \frac{1}{z}\,\mathbf{k}$$

5. Use the parametrization ($x = a\sin\phi\cos\theta, y = a\sin\phi\sin\theta, z = a\cos\phi$) for the sphere $x^2+y^2+z^2=a^2$ to find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere in the direction away from the origin. $\boxed{\text{E}}$

[提示：球的表面積是 $4\pi a^2$。]

6. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$. $\boxed{\text{F}}$

7. 以下敘述，哪些是正確的？ $\boxed{\text{G}}$

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

(b) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ converges absolutely.

(c) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ converges conditionally.

(d) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is convergent.

(e) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ is also convergent.

8. Find the interval of convergence for the following power series. $\boxed{\text{H}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n\, 2^n}$$

9. Find the Taylor series generated by $f(x) = 1/x^2$ at $x = 1$. $\boxed{\text{I}}$

10. Find the Taylor series at $x = 0$ for the function $\cos^2 x$. $\boxed{\text{J}}$

[提示：半角公式。]

(背面也有)

以下為計算或問答題，請在考試卷上盡量依序作答，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Use Green's Theorem to find the counterclockwise circulation, and outward flux, for the field $\mathbf{F} = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$ along or across the curve C: The right-hand loop of the lemniscate $r^2 = \cos 2\theta$.

2. (*15* points) (1) Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell S: $4x^2 + y + z^2 = 4$, $y \geq 0$. Let

$$\mathbf{F} = (-z + \frac{1}{2+x})\,\mathbf{i} + \tan^{-1} y\,\mathbf{j} + (x + \frac{1}{4+z})\,\mathbf{k}.$$

Find the flux of $\nabla \times \mathbf{F}$ across S:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

(2) 令 **F** 定義如上，以下敘述是否正確？為甚麼？

Let D be the region enclosed by the surface S. By the Divergence Theorem, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot (\nabla \times \mathbf{F})\, dV.$$

Since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, so

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0.$$

3. (*10* points) Let

$$a_n = \begin{cases} n/2^n & n \text{ odd} \\ 1/2^n & n \text{ even.} \end{cases}$$

Does $\sum_{n=1}^{\infty} a_n$ converge? Give reasons for your answer.

(背面還有)

4. (*10* points) Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$ with counterclockwise orientation viewing from above. Let $|R|$ be the area of the region enclosed by C in the plane. Write

$$\oint_C 2y\,dx - 3z\,dy + x\,dz$$

in terms of $|R|$.

5. (*10* points) Use the binomial series for $(1-x^2)^{-1/2}$ and the fact that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

to show that (formally)

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \cdot \frac{x^{2n+1}}{2n+1}$$

and to determine the radius of convergence.

(背面也有)