微積分聯合教學　　先修班 段考一　　Oct 21, 2008

考試時間 120 分鐘，試題共十二題，滿分 120 分。請在考試卷上以中文或英文盡量依序作答，請詳列計算過程，否則不予計分。需標明題號但不必抄題。考試卷務必寫學號、姓名，試題不必繳回。

1. (*10* points) Which of the following series converge, and which diverge? Give reasons for your answers.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$　　**b.** $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$　　**c.** $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

2. (*10* points) Which of the following series converge absolutely, which converge conditioanlly? Give reasons for your answers. [可以引用第 1 題的結果。]

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^2}$　　**b.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+4n+1}$　　**c.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{1+n^2}$

3. (*10* points) Use the binomial series and the fact that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

to generate the first four nonzero terms of the Taylor series for $\sin^{-1} x$. What is the radius of convergence?

4. (*10* points) Use series to evaluate the following limits.

a. $\displaystyle\lim_{y \to 0} \frac{y - \tan^{-1} y}{y^3}$　　**b.** $\displaystyle\lim_{x \to \infty} x^2 (e^{-1/x^2} - 1)$

5. (*10* points) Evaluate the integral

$$\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$$

6. (*10* points) Convert the following parametric equations into Cartesian equations, and sketch the graph of the equations.

a. $x = 4\cos t, \quad y = 5\sin t; \quad 0 \le t \le \pi$

b. $x = t, \quad y = \sqrt{t}; \quad t \ge 0$

c. $x = -\sec t, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

7. (*10* points) 請寫出符合以下條件的橢圓方程式。長軸在 x 軸上，右邊的焦點在原點，長軸的長度是 12，右邊的準線是 $x = 8$。也請指出：這個橢圓的另一個焦點在哪裡？離心率是多少？橢圓上任一點到兩個焦點距離的和是多少？

8. (*10* points) Let $p > 1$, determine if the following series is convergent or divergent. Give reasons and show your work.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

9. (*10* points) Find the Taylor series for

a. $\dfrac{1}{x^2}$ at $x = 1$

b. $\dfrac{1}{(1+x^2)^3}$ at $x = 0$

10. (*10* points) Write down the Taylor series for $\cos x$ at $x = 0$, and estimate the definite integral

$$\int_0^1 \cos x^2 \, dx$$

with an error of magnitude no greater than 0.001.

11. (*10* points) **(a)** State the definition of

$$\lim_{n\to\infty} a_n = L$$

(b) State the definition of

$$\sum_{n=1}^{\infty} a_n = L$$

12. (*10* points) Let $a_n > 0$, $0 \le \rho < 1$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.