微積分學科教學  先修班 段考二    Dec 2, 2008

考試時間 120 分鐘，試題共十二題，滿分 120 分。賭在考覈卷上以中文或英文
盡量依序作答，請詳列計算過程，否則不予計分。題標明題號但不必抄
題。考覈卷務必寫學號、姓名，試題不必繳回。

1. (10 points) Find the area of the surface generated by revolving the curve \( r^2 = \cos 2\theta \)
   about the x-axis.

2. (10 points) Which of the following are not always true? For each case, give a
counter-example.
   a. \( |u| = \sqrt{u \cdot u} \)  b. \( u \cdot u = |u| \)  c. \( u \times 0 = 0 \)  d. \( u \times (-u) = 0 \)
   e. \( u \times v = v \times u \)  f. \( u \times (v + w) = u \times v + u \times w \)
   g. \( (u \times v) \cdot v = 0 \)  h. \( (u \times v) \cdot w = u \cdot (v \times w) \)

3. (10 points) Find the curvature \( \kappa \) of the helix \( \mathbf{r} = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k} \) where
   \( a, b > 0 \). What is the largest value \( \kappa \) can have for a given value of \( b \)?

4. (10 points) An object of mass \( m \) travels along the parabola \( y = x^2 \) with a constant
   speed of 10 units/sec. What is the force on the object due to its acceleration at
   \((0, 0)\) and at \((\sqrt{2}, 2)\)? Write your answers in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

5. (10 points) Find the value of \( \partial z / \partial x \) at the point \((1, 1, 1)\) if the equation

   \[ xy + z^3 x - 2yz = 0 \]

   defines \( z \) as a function of the two independent variables \( x \) and \( y \). Then find \( \partial x / \partial z \) at
   the same point if the same equation defines \( x \) as a function of the two independent
   variables \( y \) and \( z \).

6. (10 points) In the following problems, find the derivative of the function at \( P \) in
   the direction of \( \mathbf{w} \):
   a. \( f(x, y) = 2xy - 3y^2, \ P(5, 5), \ \mathbf{w} = 4\mathbf{i} + 3\mathbf{j} \).
   b. \( g(x, y, z) = \cos xy + e^{yz} + \ln xz, \ P(1, 0, \frac{1}{2}), \ \mathbf{w} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \).

7. (10 points) Find the linearization \( L(x, y) \) of the function

   \[ f(x, y) = x^2 - 3xy + 5 \]
at $P(2,1)$. Then find an upper bound for the magnitude $|E(x, y)|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle

$$R : \quad |x - 2| \leq 0.1, \quad |y - 1| \leq 0.1.$$ 

The error satisfies

$$|E(x, y)| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2$$

if $M$ is any upper bound for the values of $|f_{xx}|$, $|f_{xy}|$ and $|f_{yy}|$ on $R$.

8. (10 points) Let $T = g(x, y)$ be the temperature at the point $(x, y)$ on the ellipse

$$x = 2\sqrt{2} \cos t, \quad y = \sqrt{2} \sin t, \quad 0 \leq t \leq 2\pi$$

and suppose that

$$\frac{\partial T}{\partial x} = y \quad \text{and} \quad \frac{\partial T}{\partial y} = x.$$

a. Locate the maximum temperatures on the ellipse by examining $dT/dt$ and $d^2T/dt^2$.

b. Suppose that $T = xy - 2$, find the maximum values of $T$ on the ellipse.

9. (10 points) Find the critical point of

$$f(x, y) = xy + 2x - \ln x^2 y$$

in the open first quadrant $(x > 0, y > 0)$ and show that $f$ takes on a minimum there. Remember that the Hessian of $f$ is

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

10. (10 points) Find the largest product $xyz$ the positive numbers $x, y$ and $z$ can have if they satisfy the equation $x + y + z^2 = 16$.

11. (10 points) 写出以下概念的定义：

   a. $f(x, y)$ is continuous at the point $(x_0, y_0)$.   

   b. $\frac{\partial f}{\partial x}\bigg|_{(x_0, y_0)}$

   c. The point $(x_0, y_0)$ is a critical point of the function $f(x, y)$ in a region $R$.

12. (10 points) In the following graphs, 0–9 show level curves for the functions graphed in a–j. Match each set of curves with the appropriate function.