1. Let \( \mathbf{v} \) and \( \mathbf{w} \) be vectors, \( \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v} \).

2. A tangent plane does not exist at saddle points of a surface.

3. Suppose \( C \) is a counterclockwise simple closed curve in the \( xy \)-plane, and \( R \) is the region bounded by \( C \). Then the area of \( R \) equals \( \int_C x \, dy \).

4. The function \( f(x, y) = x^3 - xy + y^3 \) has a saddle point at \( \left( \frac{1}{3}, \frac{1}{3} \right) \).
5. The series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \sqrt{n + 1}} \) converges absolutely.

Applying the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \cdot 2^n \sqrt{n+2}}{(-1)^n \cdot 2^n \sqrt{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\sqrt{n+2}}{\sqrt{n+1}} \right| = \frac{1}{2} < 1
\]

So the series converges absolutely.

6. Define

\[
f(x, y) = \begin{cases} \frac{y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0), \end{cases}
\]

then \( f(x, y) \) is continuous at \((0, 0)\).

Observe along \(x\)-axis:

\[
\lim_{(x, y) \to (0, 0), y \neq 0} f(x, y) = \lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \frac{0}{x^2} = 0
\]

Observe along \(y\)-axis:

\[
\lim_{(x, y) \to (0, 0), x \neq 0} f(x, y) = \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{y^2}{y^2} = 1
\]

Since \(0 \neq 1\), \(\lim_{(x, y) \to (0, 0)} f(x, y)\) does not exist. In particular, \(\lim_{(x, y) \to (0, 0)} f(x, y) \neq f(0, 0)\). So \(f(x, y)\) is not continuous at \((0,0)\).

7. The outward-pointing unit normal to the sphere of the radius 3 centered at the origin at \(P = (2, 2, 1)\) is \(\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)\).

The outward-pointing normal to the sphere of the radius 3 centered at the origin is

\[
\langle \cos \phi \sin \theta, \sin \phi \sin \theta, \cos \phi \rangle
\]

\[
\therefore \cos \phi = \frac{1}{3} \Rightarrow \sin \phi = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{\sqrt{8}}{3}
\]

\[
\cos \theta = \frac{2}{3} \Rightarrow \sin \theta = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{1}{3}
\]

\[
\therefore \langle \cos \phi \sin \theta, \sin \phi \sin \theta, \cos \phi \rangle = \langle \frac{2}{3}, \frac{\sqrt{8}}{3}, \frac{1}{3} \rangle
\]
8. Suppose that the maximum of \( f(x,y) \) subject to the constraint \( g(x,y) = 0 \) occurs at a point \( P = (a,b) \) such that \( \nabla f_p \neq 0 \), then \( \nabla f_p \) is tangent to \( g(x,y) = 0 \) at \( P \).

By Theorem 3 (in textbook, page 843), \( \nabla f_p \) and \( \nabla g_p \) are parallel vectors. The gradient \( \nabla g_p \) is orthogonal to \( g(x,y) = 0 \) at \( P \), hence \( \nabla f_p \) is also orthogonal to \( g(x,y) = 0 \) at \( P \).

9. The scalar line integral does not depend on how the curve to be parametrized.

10. The double integral \( \int_{0}^{2\pi} \int_{0}^{1} r^3 \, dr \, d\theta \) represents the integral of \( x^2 + y^2 \) over the unit circle.
1. If $\lim_{n \to \infty} a_n \sqrt{n} = 3$, then find $\lim_{n \to \infty} a_n$. Answer: \[ 0 \]

\[
\begin{align*}
\lim_{n \to \infty} a_n &= \lim_{n \to \infty} \left[ (a_n \sqrt{n}) \sqrt{n} \right] \\
&= \left[ \lim_{n \to \infty} (a_n \sqrt{n}) \right] \left[ \lim_{n \to \infty} \sqrt{n} \right] \\
&= 3 \cdot 0 \\
&= 0
\end{align*}
\]

2. Assume that $f(2, 3) = 8$, $f_x(2, 3) = 5$, and $f_y(2, 3) = 7$. Estimate $f(2, 3.1)$.

Answer: \[ 8.7 \]

The linear approximation at $(2, 3)$ is

\[
\begin{align*}
f(2+h, 3+k) &\approx f(2, 3) + f_x(2, 3) h + f_y(2, 3) k \\
&\approx 8 + 5h + 7k
\end{align*}
\]

\[
\Rightarrow f(2, 3.1) \approx 8 + 5 \times 0 + 7 \times 0.1
\]

\[
\approx 8.7
\]
3. Evaluate \( \int_0^1 \int_0^{\sqrt{y}} \frac{\sin x}{x^2} \, dx \, dy \). Answer: \( 1 - \cos 1 \).

原式: \( \int_0^1 \int_0^{\sqrt{y}} \frac{\sin x}{x^2} \, dx \, dy \)

\[ = \int_0^1 \sin x \, dx \]

\[ = 1 - \cos 1 \]

4. Evaluate the line integral \( \int_C (2y - e^{\cos x}) \, dx + (5x + \sqrt{y^4 + 2}) \, dy \), where \( C \) is the circle \( x^2 + y^2 = 9 \) oriented counterclockwise. Answer: \( 27 \pi \).

Let \( P = 2y - e^{\cos x} \), \( Q = 5x + \sqrt{y^4 + 2} \)

By Green's theorem:

\[ \oint_C P \, dx + Q \, dy = \iint_R (Q_x - P_y) \, dA, \] where \( R \) is the region enclosed by \( C \)

\[ \Rightarrow \int_C (2y - e^{\cos x}) \, dx + (5x + \sqrt{y^4 + 2}) \, dy \]

\[ = \iint (5 - 2) \, dA \]

\[ = 3 \iint 1 \, dA = 3 \times (9 \pi) = 27 \pi \]
5. Set up the integral with the order $d\rho d\phi d\theta$ for evaluating the volume of the solid bounded below by the sphere $\rho = 2 \cos \phi$ and above by the cone $z = \sqrt{x^2 + y^2}$.

Answer: \[
\iiint \rho^2 \sin \phi \, d\rho d\phi d\theta
\]

(Do not evaluate the integral)

\[
p = 2 \cos \phi \quad (x^2 + y^2 + (z-1)^2 = 1)
\]

\[
z = \sqrt{x^2 + y^2} \quad (\phi = \frac{\pi}{4})
\]

6. Let $\mathbf{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$ be the vortex vector field. Compute $\int_{C_R} \mathbf{F} \cdot ds$, where $C_R$ is the circle of radius $R$ centered at the origin oriented counterclockwise.

Answer: \[2\pi\]

$C_R$ can be parametrized by $C(t) = \langle R \cos t, R \sin t \rangle$, $0 \leq t \leq 2\pi$, $R > 0$

$\Rightarrow C'(t) = R \langle -\sin t, \cos t \rangle$ and $F(C(t)) = \frac{1}{R} \langle -\sin t, \cos t \rangle$

$\Rightarrow F(C(t)) \cdot C'(t) = 1$

$\Rightarrow \int_{C} F \cdot ds = \int_{0}^{2\pi} F(C(t)) \cdot C'(t) \, dt$

$= \int_{0}^{2\pi} 1 \cdot dt = 2\pi$
7. Evaluate \( \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx \). Answer: \( 2\pi \)

Let \( x = r \cos \theta, \ y = r \sin \theta \)

\[
\begin{vmatrix}
x(x, y) & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} 
\end{vmatrix} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
r \sin \theta & r \cos \theta 
\end{vmatrix} = r
\]

\[
\Rightarrow \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx \\
= \int_0^2 \int_0^{\frac{\pi}{2}} r^2 \cdot r \, d\theta \, dr \\
= \int_0^2 \left( r^3 \theta \right)_0^{\frac{\pi}{2}} \, dr \\
= \frac{\pi}{2} \int_0^2 r^3 \, dr = \frac{\pi}{2} \times 4 = 2\pi
\]

8. Evaluate \( \iiint_W f(x, y, z) \, dV \) for the function \( f(x, y, z) = x + y \) and the region \( W: y \leq z \leq x, 0 \leq y \leq x, 0 \leq z \leq 1 \). Answer: \( \frac{1}{6} \)

\[
\iiint_W f(x, y, z) \, dV = \int_0^1 \int_0^x \int_y^x (x + y) \, dz \, dy \, dx \\
= \int_0^1 \int_0^x (x^2 - y^2) \, dy \, dx \\
= \int_0^1 (x^3 - \frac{x^3}{3}) \, dx \\
= \frac{2}{3} \int_0^1 x^3 \, dx \\
= \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}
\]
1. (10 points) Compute the surface integral of \( \mathbf{F} = (0,0,z^2) \) over the oriented surface given by

\[ \phi(u,v) = (u \cos v, u \sin v, v), \]

\[ 0 \leq u \leq 1, 0 \leq v \leq 2\pi, \]

using upward-pointing normal.

\[ \frac{\partial \phi}{\partial u} = \langle \cos v, \sin v, 0 \rangle \]
\[ \frac{\partial \phi}{\partial v} = \langle -u \sin v, u \cos v, 1 \rangle \]

\[ \mathbf{n}(u,v) = \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} = \langle \sin v, -\cos v, u \rangle \]

\[ \int_S \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^1 \mathbf{F}(\phi(u,v)) \cdot \mathbf{n}(u,v) \, du \, dv \]

\[ = \int_0^{2\pi} \int_0^1 0 \cdot 0 \cdot \sin v, -\cos v, u \rangle \, du \, dv \]

\[ = \int_0^{2\pi} \int_0^1 uv^2 \, du \, dv \]

\[ = \int_0^{2\pi} \int_0^1 \frac{v^2}{2} \, du \, dv \]

\[ = \frac{1}{6} (2\pi)^3 \]

\[ = \frac{4\pi^3}{3} \]
2. (10 points) Find the area of the portion of the surface \( z = xy \) that lies within the cylinder \( x^2 + y^2 = 4 \).

The surface \( S \) can be parametrized by \( \Phi(x, y) = <x, y, xy> \), \( x^2 + y^2 \leq 4 \).

\[
\text{Area}(S) = \iint_S 1 \, ds
\]

\[
= \iint_{x^2+y^2\leq 4} \sqrt{1+g_x^2+g_y^2} \, dA , \quad \text{where} \quad g(x, y) = xy
\]

\[
= \iint_{x^2+y^2\leq 4} \sqrt{1+x^2+y^2} \, dA
\]

\[
= \int_0^2 \int_0^{2\pi} \sqrt{1+r^2} r \, d\theta \, dr
\]

\[
= (\int_0^{2\pi} 1 \, d\theta)(\int_0^{2\pi} r\sqrt{1+r^2} \, dr)
\]

\[
= 2\pi \cdot \left( \frac{1}{2} (1+r^2)^{3/2} \right|_0^1
\]

\[
= \frac{2\pi}{3} \left( 5\sqrt{5} - 1 \right)
\]
3. (10 points) Use the map

\[ \Phi(u, v) = \left( \frac{u + v}{2}, \frac{u - v}{2} \right) \]

to compute \( \int \int_R \left( (x - y) \sin (x + y) \right)^2 \, dx \, dy \) where \( R \) is the square with the vertices \((\pi, 0), (2\pi, \pi), (\pi, 2\pi)\), and \((0, \pi)\).

\[
X = \frac{u + v}{2} \quad \Rightarrow \quad U = X + Y \\
Y = \frac{u - v}{2} \quad \Rightarrow \quad V = X - Y
\]

\[
\text{Jacobian:} \quad J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad |J| = \frac{1}{2}
\]

\[
\int \int_R \left( (x - y) \sin (x + y) \right)^2 \, dx \, dy = \int_0^{2\pi} \int_0^{2\pi} \left( u \sin u \right)^2 \cdot \frac{1}{2} \, du \, dv
\]

\[
= \frac{1}{4} \int_0^{2\pi} \int_0^{2\pi} u \sin^2 u \, du \, dv
\]

\[
= \frac{\pi^3}{2} \int_0^{2\pi} \sin^2 u \, du
\]

\[
= \frac{\pi^3}{3} \int_0^{2\pi} \frac{1 - \cos 2u}{2} \, du
\]

\[
= \frac{\pi^3}{3} \left[ \frac{1}{2} \sin u \right]_0^{2\pi} - \frac{1}{4} \left( \sin 2u \right) \bigg|_0^{2\pi}
\]

\[
= \frac{\pi^4}{3}
\]
4. (10 points)

a. State the definition of a conservative vector field $F = \langle F_1, F_2 \rangle$.

b. Let $F = (2xy + y^2 + 2, x^2 + 3xy^2 + 2y)$. Is $F$ conservative? (State your reasons and show your work)

c. Let $C$ be an oriented curve and let $T$ denote the unit tangent vector pointing in the forward direction along $C$. Evaluate the line integral $\int_C F \cdot T \, ds$ along the line segment from $(0, 0)$ to $(1, 1)$.

a. Some vector fields possess a special property called path independence. By this, we mean that when the vector field is integrated along a path from $P$ to $Q$, the result depends only on the endpoints $P$ and $Q$, not on the path followed. A vector field with this property is called conservative.

b. Let $F_1 = 2xy + y^2 + 2, F_2 = x^2 + 3xy^2 + 2y$.

\[ \frac{\partial F_1}{\partial y} = 2x + 3y^2, \quad \frac{\partial F_2}{\partial x} = 2x + 3y^2 \]

\[ \Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \]

\[ \therefore F \text{ is conservative.} \]

c. By Theorem 4 (in textbook page 950), there exists a potential function $\Phi$ such that

\[ \frac{\partial \Phi}{\partial x} = F_1 \text{ and } \frac{\partial \Phi}{\partial y} = F_2. \]

\[ \frac{\partial \Phi}{\partial x} = 2xy + y^2 + 2 \Rightarrow \Phi(x, y) = x^2y + xy^3 + 2x + g(y) \]

$F_2 = \frac{\partial \Phi}{\partial y} = x^2 + 3xy^2 + g'(y)$. Since $F_2 = x^2 + 3xy^2 + 2y$, we obtain $g'(y) = 2y \Rightarrow g(y) = y^2 + c$.

Therefore, $\Phi(x, y) = x^2y + xy^3 + 2x + y^2 + C$

\[ \int_c F \cdot T \, ds = \Phi(1, 1) - \Phi(0, 0) = 5 \]
5. (10 points) Find the extreme values of the function

\[ f(x, y) = x^3 + y^3 - 3xy \]

on the given domain, \(0 \leq x, y \leq 1\).

\[ \begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 - 3y \\
\frac{\partial f}{\partial y} &= 3y^2 - 3x \\
\end{align*} \]

\[(\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad \text{critical points: } (0, 0), (1, 1) \]
\[f(0, 0) = 0, \quad f(1, 1) = -1 \]

II. \textbf{On the boundary:}

1. \textbf{OA: } \( y = 0 \), \( 0 \leq x \leq 1 \) \( \Rightarrow \) \( f(x, y) = x^3 \)
   \[\text{Max}: f(1, 0) = 1; \quad \text{Min}: f(0, 0) = 0 \]

2. \textbf{AB: } \( x = 1 \), \( 0 \leq y \leq 1 \) \( \Rightarrow \) \( f(x, y) = 1 + y^3 - 3y \)
   \[\text{Max}: f(1, 0) = 1; \quad \text{Min}: f(1, 1) = -1 \]

3. \textbf{BC: } \( y = 1 \), \( 0 \leq x \leq 1 \) \( \Rightarrow \) \( f(x, y) = x^3 + 1 - 3x \)
   \[\text{Max}: f(0, 1) = 1; \quad \text{Min}: f(1, 1) = -1 \]

4. \textbf{CD: } \( x = 0 \), \( 0 \leq y \leq 1 \) \( \Rightarrow \) \( f(x, y) = y^3 \)
   \[\text{Max}: f(0, 1) = 1; \quad \text{Min}: f(0, 0) = 0 \]

By I and II, we conclude that the global maximum is \( f(1, 0) = f(0, 1) = 1 \) and the global minimum is \( f(1, 1) = -1 \).

(試題結束)