微積分聯合教學先修班　期末考　Jan 12, 2010

考試時間 120 分鐘，題目卷有兩張紙，共三面，滿分 120 分。所有題目的答案都請按題號依序寫在答案卷，而是非與填充題必須寫在第一頁。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘內不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

是非題 (*30* points)，請答 **T** (True) 或 **F** (False)。每題 5 分。請按題號依序寫在答案卷第一頁上。

1. The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ along a path from P to Q does not depend on which path $\mathcal{C}$ is chosen.

2. If $\mathbf{F}(P) = \mathbf{e_n}(p)$ at each point P on an orientable surface $\mathcal{S}$, then $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ is equal to the area of $\mathcal{S}$. Here $\mathbf{e_n}(P)$ is the unit normal vector of $\mathcal{S}$ at P determined by the orientation.

3. The arc length parametrization of $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 3\rangle$ is

$$\mathbf{r}_1(s) = \langle 3\cos\frac{s}{3}, 3\sin\frac{s}{3}, 3\rangle.$$

4. The infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ is convergent because

$$\frac{1}{\sqrt{n^2+1}} < \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0.$$

5. If $f(x, y)$ is differentiable and $P(x_0, y_0)$ is *not* a critical point of f, then f must be increasing in the direction of $\nabla f(x_0, y_0)$ at P.

6. By Fubini's Theorem,

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \frac{x^2-y^2}{x^2+y^2}\,dy\,dx.$$

填充題 (*50* points)，[A]–[J] 每格 5 分。請按題號依序寫在答案卷第一頁上。

1. Let $\phi(x, y, z) = xy\sin yz$. Evaluate $\int_{\mathcal{C}} \nabla\phi \cdot d\mathbf{s}$, where $\mathcal{C}$ is any path from $(0, 0, 0)$ to $(1, 1, \pi)$. Answer: [A]

2. Calculate the surface integral $\iint_{\mathcal{S}} \sqrt{x^2+y^2}\, dS$ where $\mathcal{S}$ is parametrized by $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$. Answer: $\boxed{\text{B}}$

3. Let $\mathbf{F} = \langle y, z, x\rangle$ and let $\mathcal{S}$ be the oriented surface parametrized by $\Phi(u,v) = (u^2 - v, u+v, v^2)$ for $0 \le u \le 1$ and $-1 \le v \le 1$.

a. Find $\mathbf{F}\cdot\mathbf{n}$ according to the given parametrization. Answer: $\boxed{\text{C}}$

b. Evaluate $\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$. Answer: $\boxed{\text{D}}$

4. Let $\mathbf{r}(t) = \langle 1+2t, t^2, \frac{1}{3}t^3\rangle$. Find $N(1)$. Answer: $\boxed{\text{E}}$

5. Write the Taylor polynomial of degree 5 with center $c = 0$ for e^{-x^2}.

Answer: $\boxed{\text{F}}$

6. Let $f(x,y,z) = e^{-x^2-y^2-z^2}$. Compute ∇f. You can use the notation for $r = \sqrt{x^2+y^2+z^2}$. Answer: $\boxed{\text{G}}$

7. Find the global maximum of $f(x,y) = x^3 + x^2y + 2y^2$ in the region of $x \ge 0$, $y \ge 0$, and $x + y \le 1$. Answer: $\boxed{\text{H}}$

8. Calculate the double integral $\int_0^1 \int_{y=x}^1 x e^{y^3}\, dy\, dx$. Answer: $\boxed{\text{I}}$

9. Calculate the integral for $f(x,y) = |xy|$ over the region $x^2 + y^2 \le 1$.

Answer: $\boxed{\text{J}}$

計算問答證明題 (*40* points)，每題 10 分。請按題號依序從答案卷第二頁開始寫。可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Let $\mathcal{S}$ be the surface parametrized by $\Phi(r,\theta) = \big(r\cos\theta, r\sin\theta, 1-r^2\big)$. Find an normal vector $\mathbf{n}(r,\theta)$ at the point $(r,\theta) = (\frac{1}{2}, \frac{\pi}{4})$. Then find the equation of the tangent plane to $\mathcal{S}$ at this point.

2. (*10* points) Let C be the line segment joining the points (x_1, y_1) and (x_2, y_2).

a. Find a parametrization of C.

b. Show that

$$\frac{1}{2}\int_C -y\, dx + x\, dy = \frac{1}{2}(x_1y_2 - x_2y_1).$$

c. Use the Green's Theorem to find the area of the polygon with vertices (x_1, y_1), (x_2, y_2), …, (x_n, y_n), and $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.

3. (*10* points) Let L be the length of a ladder that can reach over a fence of height h to a wall located a distance b behind the fence, as in the figure. That is, $L = f(x, y) = (x+b)^2 + (y+h)^2$. Determine the constraint of this problem and find the minimum of L in terms of the positive constants h and b.

4. (*10* points) Calculate

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\, dA$$

where $\mathcal{D}$ is the interior of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$.

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