先修班　段考三　**Aug 12, 2010**

考試時間 120 分鐘，題目卷爲兩張紙，共兩頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

計算問答證明題(*120* points)，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*10* points) Evaluate the following limits:

a. $\lim_{x\to\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx})$.

b. $\lim_{x\to-\infty}(x^2+x^3)$.

2. (*10* points) Find a curve $y=f(x)$ with the following properties:

(1) **a.** $y''=2+\cos(x)$.

b. Its graph passes through the point $(0,1)$.

c. The slope of the tangent line to the curve at $(0,1)$ is equal to 4.

(2)Find an equation $xy=x^2+4$ of the slant asymptote.

3. (*10* points) Evaluate the integral, if it exists.

(1) $\int_0^{\pi/4}(1+\tan t)^3\sec^2 t dt$.

(2) $\int_0^a x\sqrt{x^2+a^2}dx.(a>0)$

4. (*20* points)

(1)State:

a. The Fundamental Calculus Theorem, Part I.

b. The Fundamental Calculus Theorem, Part II.

(2)Evaluate:

a. $\int_0^{\pi/2}\frac{d}{dx}(\sin\frac{x}{2}\cos\frac{x}{3})dx$.

b. $\frac{d}{dx}\int_0^{\pi/2}\sin\frac{x}{2}\cos\frac{x}{3}dx$.

c. $\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}dt.$

5. (*10* points) If $x\sin\pi x = \int_0^{x^2} f(t)dt$, where f is a continuous function, find $f(4)$.

6. (*10* points) Sketch the region enclosed by the given curve $x = 1 - y^2, x = y^2 - 1$. Decide whether to integrate with respect to x or y. Then find the area of the region.

7. (*10* points) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, $y = x^3, y = x, x \geq 0$; about the x-axis.

8. (*10* points) A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.

9. (*10* points) Evaluate $\lim_{n\to\infty}(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + ... + \frac{1}{\sqrt{n}\sqrt{n+n}})$.

10. (*20* points) Let $f(x) = \frac{2x^2}{x^2-1}$.

(1) Determine the domain of $f(x)$.

(2) Calculate $f'(x)$, and determine the critical numbers of $f(x)$.

(3) Determine the intervals on which $f(x)$ increases/decreases.

(4) Calculate $f''(x)$, and determine the intervals on which $f(x)$ is concave downward/upward.

(5) Determine the points of inflection.

(6) Sketch the curve $y = \frac{2x^2}{x^2-1}$.

(試題結束)

99學年暑期微積分先修班第三次段考參考解答

1. (a) 原式 $= \lim_{x\to\infty} \dfrac{ax-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \lim_{x\to\infty} \dfrac{a-b}{\sqrt{1+\frac{a}{x}}+\sqrt{1+\frac{b}{x}}} = a-b$

   (b) 原式 $= \lim_{x\to-\infty} x^3(\dfrac{1}{x}+1) \to -\infty$

2. (1) $f(x) = x^2 - \cos x + 4x + 2$

   (2) slant aymptote : $x = y$, vertical asymptote : $y = 0$

3. (1) 原式 $= \displaystyle\int_1^2 u^3 du = \frac{1}{4}u^4|_1^2 = \frac{15}{4}$

   (2) 原式 $= \displaystyle\int_a^{\sqrt{2}a} \frac{1}{2}u^{\frac{1}{2}} du = \frac{1}{3}u^{\frac{3}{2}}|_a^{\sqrt{2}a} = \frac{1}{3}\left((\sqrt{2}a)^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

4. (1) Suppose $f(x)$ is continuous on $[a,b]$.
   Part I : If $g(x) = \displaystyle\int_a^x f(t)dt$ on $[a,b]$, then $g'(x) = f(x)$ on $[a,b]$.
   Part II : $\displaystyle\int_a^b f(t)dt = F(b) - F(a)$ where F is an antiderivative of f.

   (2) (a) $\frac{\sqrt{6}}{4}$ (b) 0 (c) $-\sin\frac{x}{2}\cos\frac{x}{3}$

5. 原式 $\Rightarrow \sin(\pi x) + \pi x\cos(\pi x) = 2xf(x^2)$
   $\therefore f(4) = \frac{1}{4}(\sin(2\pi) + 2\pi\cos(2\pi)) = \frac{\pi}{2}$

6. area $= \displaystyle\int_{-1}^1 \left[(1-y^2)-(y^2-1)\right] dy = \frac{8}{3}$

7. volume $= \displaystyle\int_0 1\pi\left[x^2 - x^6\right] dx = \pi\left(\frac{1}{3}x^3 - \frac{1}{7}x^7\right)\Big|_0^1 = \frac{4}{21}\pi$

8. $\dfrac{\pi}{9}\sqrt{\dfrac{2}{3}}R^3$

9. 原式 $= \displaystyle\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{\sqrt{1+i\cdot\frac{1}{n}}}\cdot\frac{1}{n} = \int_0^1 \frac{1}{\sqrt{1+x}}dx = 2\cdot\sqrt{1+x}\Big|_{x=0}^1 = 2\left(\sqrt{2}-1\right)$

10. (1) $\{x\in\mathbb{R} | x \neq -1, 1\}$

    (2) $f'(x) = \dfrac{-4x}{(x^2-1)^2}$, crtical number : $x = 0$

    (3) $f(x)$ is increasing on $(-\infty,-1)\cup(-1,0)$; $f(x)$ is decreasing on $(0,1)\cup(1,\infty)$

(4) $f''(x) = \dfrac{4x^2+4}{(x^2-1)^3}$; convave upward : $(-\infty, -1) \cup (1, \infty)$; concave downward : $(-1, 1)$

(5) No inflection point.

(6) (略)