1. (20 points)
   (1) Let \( f(x) = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}, x > 0 \).
   (a) Prove \( f(x) \) is 1-1. (5pt.)
   (b) Find \( f^{-1} \). (5pt.)
   (c) Find domain and range of \( f^{-1} \). (5pt.)
   (2) Find \( (f^{-1})'(a) \) with \( f(x) = 3 + x^2 + \tan(\pi x/2), -1 < x < 1, a = 3 \). (5pt.)

2. (10 points)
   (1) Show that \( \sqrt{1 + x} < 1 + \frac{x}{2} \) if \( x > 0 \). (5pt.)
   (2) Find the average value of the function \( f(t) = t \sin(t^2) \) on the interval \([0, 10] \). (5pt.)

3. (10 points) Evaluate the following integrals:
   (1) \( \int_{-1}^{1} x^8 \sin x \, dx \). (5pt.)
   (2) \( \int \cos x \ln (\sin x) \, dx \). (5pt.)

4. (10 points) Find an equation of the tangent line to the curve \( xe^y + ye^x = 1 \) at the point \((0, 1)\).
5. (10 points) Evaluate the following integrals:
   (1) \[ \int \frac{\log_{10} x}{x} \, dx \] (5pt.)
   (2) \[ \int x^2e^x \, dx \] (5pt.)

6. (10 points)
   (1) Find the limit \[ \lim_{x \to \infty} x^{1/x} \]. (Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.) (5pt.)
   (2) Evaluate \[ \lim_{x \to 0} \frac{1}{x^3} \int_0^x \sin(t^2) \, dt \]. (5pt.)

7. (10 points)
   (1) Simplify the expression of \( \tan(\sin^{-1} x) \). (5pt.)
   (2) Prove \[ \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1 + x^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \]. (5pt.)

8. (10 points) Find \[ \int \frac{4x}{x^3 - x^2 - x + 1} \, dx \].

9. (20 points) Let \( y = \tan x, \ y = x, \ x = \frac{\pi}{3} \); about the \( y \)-axis.
   (a) Sketch the region bounded by the given curves. (5pt.)
   (b) Use the method of volumes by disk or washer to set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. (5pt.)
   (c) Use the method of volumes by cylindrical shell to set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. (5pt.)
   (d) Evaluate one of the integrals (a) or (b) you set up. (5pt.)

10. (10 points)
    (1) Differentiate the function \( y = \ln(x^4 \sin^2 x) \). (5pt.)
    (2) Use logarithmic differentiation to find the derivative of the function
        \[ y = \frac{(x^3 + 1)^{10} \cos^2 x}{3\sqrt{x + x^2}} \]. (5pt.)

(試題結束)
99學年暑期微積分先修班第四次段考參考解答

1. (1) (a) \( f(x) = -1 + \frac{2}{1 + \sqrt{x}} \), and \( f'(x) = \frac{1}{2} \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} \) \( < 0 \) for \( x > 0 \)
   \( \Rightarrow f(x) \) is strictly decreasing for \( x > 0 \) \( : f(x) \) is one-to-one.
   (b) \( f^{-1}(x) = \left( \frac{2}{x + 1} - 1 \right)^2 \)
   (c) domain : \(-1 < x < 1\); range : \( x > 0 \)
   (2) \( (f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{2}{\pi} \)

2. (1) Let \( f(x) = 1 + \frac{2}{x} - \sqrt{1 + x} \), then \( f'(x) = \frac{1}{2} \frac{1}{\sqrt{1 + x}} \cdot \frac{1}{1 + x} \).
   \( f'(x) > 0 \) for \( x > 0 \) and \( f(0) = 0 \), implies that,
   for each \( x > 0, \exists \xi, 0 < \xi < x \), such that \( f(x) - f(0) = f'(\xi)(x - 0) \) by Mean Value Theorem .
   \( \Rightarrow f(x) > f(0) \) for \( x > 0 \), i.e. \( \sqrt{1 + x} < 1 + \frac{2}{x} \) if \( x > 0 \).
   (2) \( \frac{1}{10} \int_{0}^{10} t \sin(t^2) dt = \frac{1}{20} (1 - \cos(100)) \)

3. (1) \( \int_{-1}^{1} x^8 \sin x dx = 0 \) (\( \sin x \) is an odd function.)
   (2) \( \sin x \ln(\sin x) - \sin x + C \)

4. \( y = -(e + 1)x + 1 \)

5. (1) 原式 = \( \int \frac{\ln x}{\ln 10} dx = \frac{1}{2} \ln 10 (\ln x)^2 + C \)
   (2) 原式 = \( \int x e^{x^2} \ln x dx = \frac{e^{x^2}}{2 \ln 2} + C \)

6. (1) 原式 = \( \lim_{x \to -\infty} e^{\frac{1}{2} \ln x} = e^{\lim_{x \to -\infty} \frac{\ln x}{x}} = e^{\lim_{x \to -\infty} \frac{1}{x} + x^2} = e^0 = 1 \)
   (2) 原式 = \( \lim_{x \to 0} \frac{\frac{d}{dx} \left( \int_{0}^{x} \sin(t^2) dt \right)}{\frac{d}{dx}(x^3)} \lim_{x \to 0} \sin(x^2) = \frac{1}{3} \)

7. (1) \( \tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}} \)
   (2) Let \( y = \tan^{-1} x \), then \( x = \tan y \) \( \Rightarrow 1 = \sec^2 y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2} \)

8. 原式 = \( \int \left( \frac{1}{x + 1} - \frac{1}{x - 1} + \frac{2}{(x - 1)^2} \right) dx = \ln \left| \frac{x + 1}{x - 1} \right| - \frac{2}{x - 1} + C \)

9. (a) (略) (注: \( \tan x > x \) for \( 0 < x < \frac{\pi}{2} \))
(b) \[ \pi \int_{0}^{\frac{\pi}{2}} (y^2 - (\tan^{-1} y)^2) \, dy + \pi \int_{\frac{\pi}{2}}^{\sqrt{3}} \left( \frac{\pi^2}{9} - (\tan^{-1} y)^2 \right) \, dy \]

(c) \[ 2\pi \int_{0}^{\frac{\pi}{2}} x(\tan x - x) \, dx \]

(d) (略)

10. (1) \[ \frac{dy}{dx} = \frac{4}{x} + \frac{2 \cos x}{\sin x} \]
(2) \[ \ln y = 10 \ln(x^3 + 1) + 2 \ln(\cos x) - \frac{1}{3} \ln(x + x^2) \]
\[ \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{10}{x^3 + 1} - 2 \tan x - \frac{2x + 1}{3(x + x^2)} \]
\[ \Rightarrow \frac{dy}{dx} = \left( \frac{10 - 3x^2}{x^3 + 1} - 2 \tan x - \frac{2x + 1}{3(x + x^2)} \right) \frac{(x^3 + 1)^{10} \cos^2 x}{\sqrt{x + x^2}} \]