先修班　期末考　　Aug 26, 2010

考試時間 120 分鐘，題目卷爲兩張紙，共兩頁，滿分 120 分。所有題目的答案都請依題號順序依序寫在答案卷上。答案卷務必寫學號、姓名，題目卷不必繳回。考試開始 30 分鐘後不得入場，開始 40 分鐘前不得離場。考試期間禁止使用字典、計算機及任何通訊器材，監試人員不得回答任何關於試題的疑問。

計算問答證明題(*120* points)，請依題號順序依序寫在答案卷上，可以用中文或英文作答。請詳列計算過程，否則不予計分。需標明題號但不必抄題。

1. (*20* points)

(1)Let $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, $x > 0$.

(a)Prove $f(x)$ is 1-1 . (5pt.)

(b)Find f^{-1}. (5pt.)

(c)Find domain and range of f^{-1}.(5pt.)

(2)Find $(f^{-1})'(a)$ with $f(x) = 3 + x^2 + \tan(\pi x/2), -1 < x < 1, a = 3$.(5pt.)

2. (*10* points)

(1)Show that $\sqrt{1+x} < 1 + \dfrac{x}{2}$, if $x > 0$.(5pt.)

(2)Find the average value of the funciton $f(t) = t\sin(t^2)$ on the interval $[0, 10]$.(5pt.)

3. (*10* points) Evaluate the following intergrals:

(1)$\displaystyle\int_{-1}^{1} x^8 \sin x \, dx$.(5pt.)

(2)$\displaystyle\int \cos x \ln(\sin x) \, dx$.(5pt.)

4. (*10* points) Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

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5. (*10* points) Evaluate the following integrals:

(1) $\displaystyle\int \frac{\log_{10} x}{x}\, dx$.(5pt.)

(2) $\displaystyle\int x2^{x^2}\, dx$.(5pt.)

6. (*10* points)

(1)Find the limit $\displaystyle\lim_{x\to\infty} x^{1/x}$. (Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.) (5pt.)

(2)Evaluate $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)\, dt$.(5pt.)

7. (*10* points)

(1)Simplify the expression of $\tan(\sin^{-1} x)$.(5pt.)

(2)Prove $\displaystyle\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2},\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.(5pt.)

8. (*10* points) Find $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\, dx$.

9. (*20* points) Let $y = \tan x$, $y = x$, $x = \dfrac{\pi}{3}$; about the y-axis.

(a)Sketch the region bounded by the given curves.(5pt.)

(b)Use the method of **volumes by disk or washer** to set up an intergral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.(5pt.)

(c)Use the method of **volumes by cylindrical shell** to set up an intergral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.(5pt.)

(d)Evaluate one of the integrals (a) or (b) you set up. (5pt.)

10. (*10* points)

(1)Differentiate the function $y = \ln(x^4 \sin^2 x)$.(5pt.)

(2)Use logarithmic differentiation to find the derivative of the function

$$y = \frac{(x^3+1)^{10}\cos^2 x}{\sqrt[3]{x+x^2}}.(5pt.)$$

(試題結束)

99學年暑期微積分先修班第四次段考參考解答

1. (1) (a) $f(x)=-1+\frac{2}{1+\sqrt{x}}$, and $f'(x)=-\dfrac{1}{2}\dfrac{1}{\sqrt{x}\,(1+\sqrt{x})^2}<0$ for $x>0$

   $\Rightarrow f(x)$ is strictly decreasing for $x>0$ $\quad\therefore f(x)$ is one-to-one.

   (b) $f^{-1}(x)=\left(\dfrac{2}{x+1}-1\right)^2$

   (c) domain : $-1<x<1$; range : $x>0$

   (2) $\left(f^{-1}\right)'(3)=\dfrac{1}{f'(f^{-1}(3))}=\dfrac{2}{\pi}$

2. (1) Let $f(x)=1+\frac{x}{2}-\sqrt{1+x}$, then $f'(x)=\dfrac{1}{2}-\dfrac{1}{2}\dfrac{1}{\sqrt{1+x}}$.

   $f'(x)>0$ for $x>0$ and $f(0)=0$, implies that,
   for each $x>0$, $\exists\xi,\ 0<\xi<x$, such that $f(x)-f(0)=f'(\xi)(x-0)$ by Mean Value Theorem .
   $\Rightarrow f(x)>f(0)$ for $x>0$, i.e. $\sqrt{1+x}<1+\frac{x}{2}$ if $x>0$.

   (2) $\dfrac{1}{10}\displaystyle\int_0^{10} t\sin(t^2)dt=\dfrac{1}{20}\left(1-\cos(100)\right)$

3. (1) $\displaystyle\int_{-1}^{1} x^8\sin x dx=0$ ($\sin x$ is an odd function.)

   (2) $\sin x\ln(\sin x)-\sin x+C$

4. $y=-(e+1)x+1$

5. (1) 原式 $=\displaystyle\int\frac{\ln x}{\ln 10}\frac{1}{x}dx=\frac{1}{2\ln 10}(\ln x)^2+C$

   (2) 原式 $=\displaystyle\int xe^{x^2\ln 2}dx=\frac{2^{x^2}}{2\ln 2}+C$

6. (1) 原式 $=\displaystyle\lim_{x\to\infty}e^{\frac{1}{x}\ln x}=e^{\lim_{x\to\infty}\frac{\ln x}{x}}=e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}}=e^0=1$

   (2) 原式 $=\displaystyle\lim_{x\to0}\frac{\frac{d}{dx}\left(\int_0^x\sin(t^2)dt\right)}{\frac{d}{dx}(x^3)}\lim_{x\to0}\frac{\sin(x^2)}{3x^2}=\frac{1}{3}$

7. (1) $\tan(\sin^{-1}x)=\dfrac{x}{\sqrt{1-x^2}}$

   (2) Let $y=\tan^{-1}x$, then $x=\tan y$ $\Rightarrow$ $1=\sec^2 y\dfrac{dy}{dx}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{\sec^2 y}=\dfrac{1}{1+x^2}$

8. 原式 $=\displaystyle\int\left(\frac{1}{x+1}-\frac{1}{x-1}+\frac{2}{(x-1)^2}dx\right)=\ln\left|\frac{x+1}{x-1}\right|-\frac{2}{x-1}+C$

9. (a) (略) (**註:** $\tan x>x$ **for** $0<x<\frac{\pi}{2}$)

(b) $\pi\int_0^{\frac{\pi}{3}}(y^2-(\tan^{-1}y)^2)dy+\pi\int_{\frac{\pi}{3}}^{\sqrt{3}}(\frac{\pi^2}{9}-(\tan^{-1}y)^2)dy$

(c) $2\pi\int_0^{\frac{\pi}{3}}x(\tan x-x)dx$

(d) (略)

10. (1) $\frac{dy}{dx}=\frac{4}{x}+\frac{2\cos x}{\sin x}$

(2) $\ln y=10\ln(x^3+1)+2\ln(\cos x)-\frac{1}{3}\ln(x+x^2)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx}=10\frac{3x^2}{x^3+1}-2\tan x-\frac{2x+1}{3(x+x^2)}$

$\Rightarrow \frac{dy}{dx}=(10\frac{3x^2}{x^3+1}-2\tan x-\frac{2x+1}{3(x+x^2)})\frac{(x^3+1)^{10}\cos^2 x}{\sqrt[3]{x+x^2}}$