CALCULUS(II) – FIRST MIDDLE EXAMINATION

1. (10 %) Find the distance between
   (a) two lines $l_1 : \mathbf{r}(t) = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ and $l_2 : \mathbf{R}(u) = (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + u(4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$;
   (b) two planes $S_1 : x + 2y + 2z = 11$ and $S_2 : 3x + 6y + 6z = 5$.

2. (10 %) Find the indicated derivative

$$\frac{d}{dt}[(e^t\mathbf{i} + e^{-t}\mathbf{j} + e^{t^2}\mathbf{k}) \cdot ((\ln t\mathbf{i} + 2t\mathbf{j} + \ln t^3\mathbf{k}) \times (\frac{1}{t}\mathbf{i} + 2\mathbf{j} + \frac{3}{t}\mathbf{k}))].$$

3. (10 %) Let $f'(t) = t\mathbf{i} + t(1+t^2)^{-\frac{1}{2}}\mathbf{j} + te^t\mathbf{k}$ and $f(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find $f(t)$.

4. (10 %) Compute the length of the Helix $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$ from $t = 0$ to $t = 2\pi$ and parametrize it by arc length. Determine also its curvature at any t.

5. (15 %)
   (a) Consider the motion of a particle with position vector $\mathbf{r}(t)$ which is constraint by a central force field $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}$. Show that the angular momentum is a constant in t.
   (b) Verify by differentiation with respect to time t and if the acceleration of a planet is given by

$$\mathbf{a} = -\rho\frac{\mathbf{r}}{r^3},$$

   then the energy $E = \frac{1}{2}mv^2 - \dfrac{m\rho}{r}$ is a constant.

6. (10 %) Analyze the continuity of the following functions at $(0, 0)$:

$$f(x,y) = \frac{xy}{x^2+y^2}; \quad f(x,y) = (x^2+y^2)\ln(x^2+y^2); \quad f(x,y) = \frac{x^2y}{x^4+y^2},$$

   where we set $f(0, 0) = 0$ in all cases.

7. (15 %) Find the partial derivatives f_x , f_y and f_{xy}.

$$f(x,y) = \ln y^x; \quad f(x,y) = 3^{xy^2}; \quad f(x,y) = y^{x^x}.$$

8. (15 %) Find an equation for the level curve of $f(x, y) = x^2 \tan^{-1} y$ that contains the point $(2, 1)$ and compute $\nabla f(2, 1)$. Show that $\nabla f(2, 1)$ is perpendicular to the level curve which passes through $(2, 1)$.

9. (10 %) The temperature in a neighborhood of the point $(\frac{1}{4}\pi, 0)$ is given by the function

$$T(x, y) = \sqrt{2}e^{-y} \cos x.$$

Find the path followed by a heat-seeking particle that originates at the center of the neighborhood.