CALCULUS(II) – SECOND MIDDLE EXAMINATION

1. (12%)

   (a) Use the chain rule to find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial s}$ if $f(x,y,z) = 3^{(x^2+y^2+2z)}$, $x = r+s$, $y = r-s$ and $z = 2rs$.

   (b) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ by implicit differentiation:

$$z^4 + x^2z^3 + y^2 + xy = 2.$$

2. (10%) Find the gradient of $f(x,y) = x^2e^{\sin y+\cos y}$ at the point $(2,\pi/2)$. Write an equation for the plane tangent to the surface $z = x^2e^{\sin y+\cos y}$ at the point $(2,\pi/2,4e)$ and a scalar parametric equation for the normal line to the surface $z = x^2e^{\sin y+\cos y}$ at the point $(2,\pi/2,4e)$.

3. (10%) Find the critical points and the local extreme values:

$$f(x,y) = \frac{-2y}{x^2+y^2+1}.$$

4. (10%) Find the absolute extreme values of $f(x,y) = y^3 - 3xy - x^3$ on the set $D = \{(x,y) : -2 \le y \le 2, y \le x \le 2\}$.

5. (10%) Use the Lagrange multiplier method to find the points on the sphere $x^2+y^2+z^2 = 1$ that are closest to and farthest from the point $(2,2,1)$.

6. (10%) Determine whether the vector function $(e^{xy}+xye^{xy})\mathbf{i}+(x^2e^{xy}+1)\mathbf{j}$ is the gradient $\nabla f(x,y)$ of a function everywhere defined. If so, find such a function.

7. (30%) Evaluate

   (a) $\displaystyle\int\int_R xe^y\ dxdy$ where $R: 0 \le x \le 1,\ 0 \le y \le x^2$;

   (b) $\displaystyle\int_0^1\int_{\sqrt{y}}^1 \cos(\frac{x^3+1}{2})\ dxdy$; (c) $\displaystyle\int_0^1\int_0^{\cos^{-1}y} e^{\sin x}\ dxdy$;

   (d) Integrate $f(x,y) = \sqrt{x^2+y^2}$ over the triangle with vertices $(0,0)$,$(1,0)$,$(1,\sqrt{3})$,.

8. (5% each)
   (a) Evaluate $\int\int_{D_b} e^{-(x^2+y^2)}dxdy$ where $D_b : x^2+y^2 \leq b^2$.
   (b) Show that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.
   (c) Compute

$$\int_{-\infty}^{\infty} \left(e^{-x^2} + xe^{-x^4}\right) dx.$$